Lesson 1.3

More Group-Ranking Models and Paradoxes

Different models for finding a group ranking can give different results. This fact led the Marquis de Condorcet to propose that a choice that could obtain a majority over every other choice should be ranked first for the group.

Again consider the set of preference schedules used in the previous lesson (see Figure 1.4).

![Figure 1.4](preferences.png)

Total number of voters:
\[8 + 5 + 6 + 7 = 26.\]

Figure 1.4. Preferences of 26 voters.

To examine these data for a Condorcet winner, compare each choice with every other choice. For example, begin by comparing A with B, then with C, and finally with D. Notice in Figure 1.4 that A is ranked higher than B on 8 schedules and lower on 18. (An easy way to see this is to cover C and D on all the schedules.) Because A cannot obtain a majority against B, A cannot be a Condorcet winner. Therefore, there is no need to check to see if A can beat C or D.
Now consider B. You have already seen that B beats A, so begin by comparing B with C. B is ranked higher than C on $8 + 5 + 7 = 20$ schedules and lower than C on 6.

Now compare B with D. B is ranked higher than D on $8 + 5 + 6 = 19$ schedules and lower than D on 7. Therefore, B has a majority over each of the other choices and so is a Condorcet winner.

Since B is a Condorcet winner, it is unnecessary to make comparisons between C and D. Although all comparisons do not always have to be made, it can be helpful to organize them in a table:

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<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>L</td>
<td>L</td>
<td>L</td>
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<td>B</td>
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<td>D</td>
<td>W</td>
<td>L</td>
<td>L</td>
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</tbody>
</table>

To see how a choice does in one-on-one contests, read across the row associated with that choice. You see that A, for example, loses in one-on-one contests with B, C, and D.

Although Condorcet’s model may sound ideal, it sometimes fails to produce a winner. Consider the set of schedules shown in Figure 1.5.

![Figure 1.5](image)

Figure 1.5. Preferences of 60 voters.

Notice that A is preferred to B on 40 of the 60 schedules but that A is preferred to C on only 20. Although C is preferred to A on 40 of the 60, C is preferred to B on only 20. Therefore there is no Condorcet winner.
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You might expect that if A is preferred to B by a majority of voters and B is preferred to C by a majority of voters, then a majority of voters prefer A to C. But the example shows that this need not be the case.

In other mathematics classes you have learned that many relationships are transitive. The relation “greater than” (>), for example, is transitive because if \(a > b\) and \(b > c\), then \(a > c\).

You have just seen that group-ranking models may violate the transitive property. Because this intransitivity seems contrary to intuition, it is known as a paradox. This particular paradox is sometimes referred to as the Condorcet paradox. There are other paradoxes that can occur with group-ranking models, as you will see in this lesson’s exercises.

Exercises

1. Find a Condorcet winner in the soft drink ballot your class conducted in Lesson 1.1.

2. Propose a method for resolving situations in which there is no Condorcet winner.

3. In a system called pairwise voting, two choices are selected and a vote taken. The loser is eliminated, and a new vote is taken in which the winner is paired against a new choice. This process continues until all choices but one have been eliminated. An example of the use of pairwise voting occurs in legislative bodies in which bills are considered two at a time. The choices in the set of preferences shown in the following figure represent three bills being considered by a legislative body.

\[
\begin{array}{ccc}
A & B & C \\
B & C & A \\
C & A & B \\
\end{array}
\]

a. Suppose you are responsible for deciding which two bills appear on the agenda first. If you strongly prefer bill C, which two bills would you place on the agenda first? Why?

b. Is it possible to order the voting so that some other choice wins? Explain.
4. A panel of sportswriters is selecting the best football team in a league, and the preferences are distributed as follows.

```
A
B
C
52
```

```
B
A
C
38
```

```
A
B
C
10
```

a. Determine a best team using a 3-2-1 Borda count.

b. The 38 who rank B first and A second decide to lie in order to improve the chances of their favorite and so rank C second. Determine the winner using a 3-2-1 Borda count.

5. When people decide to vote differently from the way they feel about the choices, they are said to be voting insincerely. People are often encouraged to vote insincerely because they have some idea of an election’s result beforehand. Explain why such advance knowledge is possible.

6. Many political elections in the United States are decided with a plurality model. Construct a set of preferences with three choices in which a plurality model could encourage insincere voting. Identify the group of voters that might be encouraged to vote insincerely and explain the effect of their insincere voting on the election outcome.

7. Many people consider plurality models flawed because they can produce a winner that a majority of voters do not like.

a. What percentage of voters ranks the plurality winner last in the preferences shown below?

```
A
C
B
D
```

```
B
D
E
```

```
D
E
C
```

```
C
B
```

```
E
A
A
A
```

```
20
18
15
12
```
b. Runoffs are sometimes used to avoid the selection of a controversial winner. Is a runoff winner an improvement over a plurality winner in this set of preferences? Explain.

c. Do you consider a sequential runoff winner an improvement over plurality and runoff winners? Explain.

8. a. Use a runoff to determine a winner in the following set of preferences.

```
<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>B</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>30</td>
<td>25</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>
```

b. In some situations, votes are made public. For example, people have the right to know how their elected officials vote on issues. Suppose these schedules represent such a situation. Because they expect to receive some favors from the winner and because they expect A to win, the seven voters associated with the last schedule decide to change their preferences from

```
<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>to</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td></td>
<td></td>
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<tr>
<td>C</td>
<td>C</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

and to “go with the winner.” Conduct a new runoff and determine the winner.

c. Explain why the results are a paradox.
Lesson 1.3 • More Group-Ranking Models and Paradoxes

9. a. Use a 4-3-2-1 Borda count to find a group ranking for the following set of preferences.

```
A  B
C  D
B  A
D  C
```

b. These preferences represent the ratings of four college athletic teams, and team C has been disqualified because of a recruiting violation. Write the schedules with team C removed and use a 3-2-1 Borda count to determine a group ranking.

c. Explain why these results are a paradox.

10. For each voting model discussed in this and the previous lesson (plurality, Borda, runoff, sequential runoff, and Condorcet), write a brief summary. Include at least one example of why the model can lead to unfair results.

11. In theory, Condorcet models require that each choice be compared with every other one, although in practice many of the comparisons do not have to be made in order to determine the winner. Consider the number of comparisons when every possible comparison is made.

Mathematicians sometimes find it helpful to represent the choices and comparisons visually. If there are only two choices, a single comparison is all that is necessary. In the diagram that follows, a point, or vertex, represents a choice, and a line segment, or edge, represents a comparison.

```
A  B
```

a. Add a third choice, C, to the diagram. Connect it to A and to B to represent the additional comparisons. How many new comparisons are there? What is the total number of comparisons?
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b. Add a fourth choice, D, to the diagram. Connect it to each of A, B, and C. How many new comparisons are there, and what is the total number of comparisons?

c. Add a fifth choice to the diagram and repeat. Then add a sixth choice and repeat. Complete the following table.

<table>
<thead>
<tr>
<th>Number of Choices</th>
<th>Number of New Comparisons</th>
<th>Total Number of Comparisons</th>
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<td></td>
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<td>6</td>
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</table>

12. Let $C_n$ represent the total number of comparisons necessary when there are $n$ choices. Write a recurrence relation that expresses the relationship between $C_n$ and $C_{n-1}$.

13. U.S. College Hockey Online (USCHO) has several ranking systems. In a system called pairwise ranking, USCHO compares each team to every other team. In each comparison, the team that compares favorably to the other is awarded a point. The team with the most points is ranked first. Consider a simple version of this system in a league with 6 teams, A, B, C, D, E, and F. The following table shows the results of the comparisons. An X in a team's row indicates that it won the comparison with the team at the top of the column.

### Top Ten Pairwise Rankings

U.S. College Hockey Online  
May 1, 2013

<table>
<thead>
<tr>
<th>Rank</th>
<th>Team</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Quinnipiac</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>Minnesota</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>Massachusetts-Lowell</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>Notre Dame</td>
<td>26</td>
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<tr>
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<td>25</td>
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<tr>
<td>5t</td>
<td>Boston College</td>
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<tr>
<td>7</td>
<td>New Hampshire</td>
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</tr>
<tr>
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<td>23</td>
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<td>9t</td>
<td>Denver</td>
<td>20</td>
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<tr>
<td>9t</td>
<td>Niagara</td>
<td>20</td>
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<table>
<thead>
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<th>B</th>
<th>C</th>
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a. Find two pairwise comparisons in this table that demonstrate the transitive property. Find two comparisons that demonstrate a violation of the transitive property.

b. If each team receives a point for each comparison that it wins, find a group ranking for these teams.

c. Suggest a modification to the point system that is advantageous to team F.

d. Suppose team D drops out of the league. What effect does this have on the rankings you found in part b?

Computer/Calculator Explorations

14. Use the preference schedule program that accompanies this book to find a set of preferences with at least four choices that demonstrates the same paradox found in Exercise 8, but when sequential runoff is used.

15. Use the preference schedule program to enter several schedules with five choices. Use the program’s features to alter your data in order to produce a set of preferences with several different winners. Can you find a set of preferences with five choices and five different winners? If so, what is the minimum number of schedules with which this can be done? Explain.

Projects

16. Research and report on paradoxes in mathematics. Try to determine whether the paradoxes have been satisfactorily resolved.

17. Research and report on paradoxes outside mathematics. In what way have these paradoxes been resolved?

18. Select an issue of current interest in your community or school that involves more than two choices. Have each member of your class vote by writing a preference schedule. Compile the preferences and determine winners by five different methods.

19. Investigate the contributions of Charles Dodgson (Lewis Carroll) to election theory. Was he responsible for any of the group-ranking procedures you have studied? What did he suggest doing when Condorcet models fail to produce a winner?
20. Investigate the system your school uses to determine academic rankings of students. Is it similar to any of the group-ranking procedures you studied? If so, could it suffer from any of the same problems? Propose another system and discuss why it might be better or worse than the one currently in use.

21. Investigate elections in your school (class officers, officers of organizations, homecoming royalty, and so forth). Report on the type of voting and the way winners are chosen. Recommend alternative methods and explain why you think the methods you recommend are fairer.

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**Switzerland Jumps to #1 as Brand US Falls Further into Decline**

Today, FutureBrand reveals its 8th annual ranking of the world’s leading country brands – moving Switzerland to the #1 seat in the consultancy’s 2012-13 Country Brand Index, a preeminent global study of country brands.

In keeping with past year’s studies, the 2012-13 CBI ranks the world’s countries – from their cultures, to their industries, to their economic vitality and public policy initiatives – based on global perceptions. Drawing insights from a 3,600 opinion-formers and frequent international travelers from 18 countries, FutureBrand used its proprietary Hierarchal Decision Model to determine how key audiences – residents, investors, tourists and foreign governments – see the world’s country brands.

**Top 25 country brands of 2012–13:**
1. Switzerland (+1 from 2012)
2. Canada (-1)
3. Japan (+1)
4. Sweden (+3)
5. New Zealand (-2)
6. Australia (-1)
7. Germany (+4)
8. United States (-2)
9. Finland (-1)
10. Norway (+2)
11. United Kingdom (+2)
12. Denmark (+3)
13. France (-4)
14. Singapore (+2)
15. Italy (-5)
16. Maldives (+2)
17. Austria (0)
18. Netherlands (+5)
19. Spain (-5)
20. Mauritius (+2)
21. Ireland (-1)
22. Iceland (-3)
23. United Arab Emirates (+2)
24. Bermuda (-3)
25. CostaRica (-1)