Lesson 1.4

Arrow's Conditions and Approval Voting

Paradoxes, unfair results, and insincere voting are some of the problems that have caused people to look for better models for reaching group decisions. In this lesson you will learn of some recent and important work that has been done in attempts to improve the group-ranking process. First, consider an example involving pairwise voting.

Ten representatives of the language clubs at Central High School are meeting to select a location for the clubs' annual joint dinner. The committee must choose among a Chinese, French, Italian, or Mexican restaurant (see Figure 1.6).

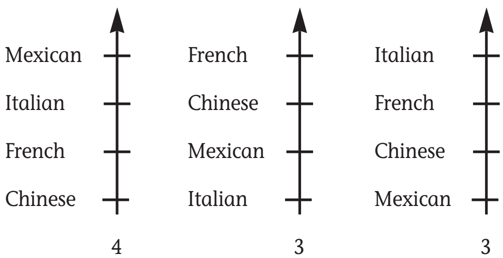


Figure 1.6. Preferences of 10 students.

Racquel says that because the last two dinners were at Mexican and Chinese restaurants, this year's dinner should be at either an Italian or a French restaurant. The group votes 7 to 3 in favor of the Italian restaurant.

Martin, who doesn't like Italian food, says that the community's newest Mexican restaurant has an outstanding reputation. He proposes that the group choose between Italian and Mexican. The other members agree and vote 7 to 3 to hold the dinner at the Mexican restaurant.

Sarah, whose parents own a Chinese restaurant, says that she can obtain a substantial discount for the event. The group votes between the Mexican and Chinese restaurants and selects the Chinese by a 6 to 4 margin.

Look carefully at the group members' preferences. Note that French food is preferred to Chinese by all, yet the voting selected the Chinese restaurant!

In 1951, paradoxes such as this led Kenneth Arrow, a U.S. economist, to formulate a list of five conditions that he considered necessary for a fair group-ranking model. These fairness conditions today are known as **Arrow's conditions**.

Mathematician of Note

Kenneth Arrow (1921–)

Kenneth Arrow received a degree in mathematics before turning to economics. His use of mathematical methods in election theory brought him worldwide recognition.

One of Arrow's conditions says that if every member of a group prefers one choice to another, then the group ranking should do the same. According to this condition, the choice of the Chinese restaurant when all members rated French food more favorably than Chinese is unfair. Thus, Arrow considers pairwise voting a flawed group-ranking method.

Arrow inspected common models for determining a group ranking for adherence to his five conditions. He also looked for new models that would meet all five. After doing so, he arrived at a surprising conclusion.

In this lesson's exercises, you will examine a number of group-ranking models for their adherence to Arrow's conditions. You will also learn Arrow's surprising result.

Arrow's Conditions

1. Nondictatorship: The preferences of a single individual should not become the group ranking without considering the preferences of the others.
2. Individual Sovereignty: Each individual should be allowed to order the choices in any way and to indicate ties.
3. Unanimity: If every individual prefers one choice to another, then the group ranking should do the same. (In other words, if every voter ranks A higher than B, then the final ranking should place A higher than B.)
4. Freedom from Irrelevant Alternatives: If a choice is removed, the order in which the others are ranked should not change. (The choice that is removed is known as an irrelevant alternative.)
5. Uniqueness of the Group Ranking: The method of producing the group ranking should give the same result whenever it is applied to a given set of preferences. The group ranking should also be transitive.

Exercises

1. Your teacher decides to order soft drinks for your class on the basis of the soft drink vote conducted in Lesson 1.1 but, in so doing, selects the preference schedule of a single student (the teacher's pet). Which of Arrow's conditions are violated by this method of determining a group ranking?
2. Instead of selecting the preference schedule of a favorite student, your teacher places all the individual preferences in a hat and draws one. If this method were repeated, would the same group ranking result? Which of Arrow's conditions does this method violate?

3. Do any of Arrow's conditions require that the voting process include a secret ballot? Is a secret ballot desirable in all group-ranking situations? Explain.

4. Examine the paradox demonstrated in Exercise 9 of Lesson 1.3 on page 23. Which of Arrow's conditions are violated?

5. Construct a set of preference schedules with three choices, A, B, and C, showing that the plurality method violates Arrow's fourth condition. In other words, construct a set of preferences in which the outcome between A and B depends on whether C is on the ballot.

6. You have seen situations in which insincere voting occurs. Do any of Arrow's conditions state that insincere voting should not be part of a fair group-ranking model? Explain.

7. Suppose that there are only two choices in a list of preferences and that the plurality method is used to decide the group ranking. Which of Arrow's conditions could be violated? Explain.

8. A group of voters have the preferences shown in the following figure.

A	A	B	C	D
C	D	C	D	B
D	B	A	B	C
B	C	D	A	A
9	5	7	4	4

a. Use plurality, Borda, runoff, sequential runoff, and Condorcet models to find winners.

b. Investigate this set of preferences for violation of Arrow's fourth condition. That is, can a choice change a winner by withdrawing?

9. Read the news article about the Google search engine.

   a. Does the transitive property apply to individual Google voting? That is, if site A casts a Google vote for site B and site B casts a Google vote for site C, then must site A cast a Google vote for site C?

   b. Does the transitive property apply to the Google ranking system? That is, if site A ranks higher than site B and site B ranks higher than site C, then must site A rank higher than site C? Explain.

10. After failing to find a group-ranking model for three or more choices that always obeyed all his fairness conditions, Arrow began to suspect that such a model does not exist. He applied logical reasoning and proved that no model, known or unknown, can always obey all five conditions. In other words, any group-ranking model violates at least one of Arrow's conditions in some situations.

    Arrow's proof demonstrates how mathematical reasoning can be applied to areas outside mathematics. This and other achievements earned Arrow the 1972 Nobel Prize in economics.

Is Google Page Rank Still Important?

Search Engine Journal
October 6, 2004

Since 1998 when Sergey Brin and Larry Page developed the Google search engine, it has relied on the Page Rank Algorithm. Google's reasoning behind this is, the higher the number of inbound links 'pointing' to a website, the more valuable that site is, in which case it would deserve a higher ranking in its search results pages. If site 'A' links to site 'B', Google calculates this as a 'vote' for site B. The higher the number of votes, the higher the overall value for site 'B'. In a perfect world, this would be true. However, over the years, some site owners and webmasters have abused the system, implementing some 'link farms' and linking to websites that have little or nothing to do with the overall theme or topic presented in their sites.

Although Arrow's work means that a perfect group-ranking model will never be devised, it does not mean that current models cannot be improved. Recent studies have led some experts to recommend **approval voting**.

In approval voting, you may vote for as many choices as you like, but you do not rank them. You mark all those of which you approve. For example, if there are five choices, you may vote for as few as none or as many as five.

a. Write a soft drink ballot like the one you used in Lesson 1.1. Place an "X" beside each of the soft drinks you find acceptable. At the direction of your instructor, collect ballots from the other members of your class. Count the number of votes for each soft drink and determine a winner.

b. Determine a complete group ranking.

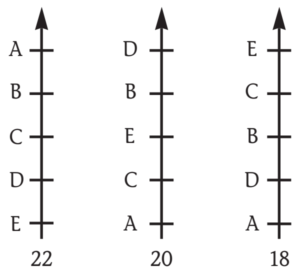
c. Is the approval winner the same as the plurality winner in your class?

d. How does the group ranking in part b compare with the Borda ranking that you found in Lesson 1.1?

11. Examine Exercise 4 of Lesson 1.3 on page 21. Would any members of the panel of sportswriters be encouraged to vote insincerely if approval voting were used? Explain.

12. What is the effect on a group ranking of casting approval votes for all choices? Of casting approval votes for none of the choices?

13. The voters whose preferences are represented below all feel strongly about their first choices but are not sure about their second and third choices. They all dislike their fourth and fifth choices. Since the voters are unsure about their second and third choices, they flip coins to decide whether to give approval votes to their second and third choices.



a. Assuming the voters' coins come up heads half the time, how many approval votes would you expect each of the five choices to get? Explain your reasoning.

b. Do the results seem unfair to you in any way? Explain.

14. Approval voting offers a voter many choices. If there are three candidates for a single office, for example, the plurality system offers the voter four choices: vote for any one of the three candidates or for none of them. Approval voting permits the voter to vote for none, any one, any two, or all three.

To investigate the number of ways in which you can vote under approval voting, consider a situation with two choices, A and B. You can represent voting for none by writing { }, voting for A by writing {A}, voting for B by writing {B}, and voting for both by writing {A, B}.

a. List all the ways of voting under an approval system when there are three choices.

b. List all the ways of voting under an approval system when there are four choices.

c. Generalize the pattern by letting V_n represent the number of ways of voting under an approval system when there are n choices and writing a recurrence relation that describes the relationship between V_n and V_{n-1}.

15. Listing all the ways of voting under the approval system can be difficult if not approached systematically. The following algorithm describes one way to find all the ways of voting for two choices. The results are shown applied to a ballot with five choices, A, B, C, D, and E.

1. List all choices in order in List 1.
2. Draw a line through the first choice in List 1 that doesn't already have a line drawn through it. Write this choice as many times in List 2 as there are choices in List 1 without lines through them.
3. Beside each item you wrote in List 2 in step 2, write a choice in List 1 that does not have a line through it.

List 1	List 2
A B C D E	
~~A~~ B C D E	A
	A
	A
	A
	A B
	A C
	A D
	A E

4. Repeat steps 2 and 3 until each choice has a line through it. The items in the second list show all the ways of voting for two items.

Write an algorithm that describes how to find all the ways of voting for three choices. You may use the results of the previous algorithm to begin the new one.

16. Many patterns can be found in the various ways of voting when the approval system is used. The following table shows the number of ways of voting for exactly one item when there are several choices on the ballot. For example, in Exercise 14, you listed all the ways of voting when there are three choices on the ballot. Three of these, {A}, {B}, and {C}, are selections of one item.

Number of Choices on the Ballot	Number of Ways of Selecting Exactly One Item
1	1
2	2
3	3
4	___
5	___

Complete the table.

17. Let $V1_n$ represent the number of ways of selecting exactly one item when there are n choices on the ballot and write a recurrence relation that expresses the relationship between $V1_n$ and $V1_{n-1}$.

18. The following table shows the number of ways of voting for exactly two items when there are from one to five choices on the ballot. For example, your list in Exercise 14 shows that when there are three choices on the ballot, there are three ways of selecting exactly two items: {A, B}, {A, C}, and {B, C}.

Number of Choices on the Ballot	Number of Ways of Selecting Exactly Two Items
2	1
3	3
4	___
5	___

Complete the table.

19. Let $V2_n$ represent the number of ways of selecting exactly two items when there are n choices on the ballot and write a recurrence relation that expresses the relationship between $V2_n$ and $V2_{n-1}$. Can you find more than one way to do this?

Computer/Calculator Explorations

20. Design a computer program that lists all possible ways of voting when approval voting is used. Use the letters A, B, C, . . . to represent the choices. The program should ask for the number of choices and then display all possible ways of voting for one choice, two choices, and so forth.

Projects

21. Investigate the number of ways of voting under the approval system for other recurrence relations (see Exercises 16 through 19). For example, in how many ways can you vote for three choices, four choices, and so forth?

22. Arrow's result is an example of an impossibility theorem. Investigate and report on other impossibility theorems.

23. Research and report on Arrow's theorem. The theorem is usually proved by an indirect method. What is an indirect method? How is it applied in Arrow's case?

24. In approval voting voters apply an approve or disapprove rating to each choice. Thus, approval voting is a rating system--not a ranking system. In 2007, Michel Balinski and Rida Laraki proposed another type of rating system called majority judgment, in which voters are allowed more than two ratings. Research and report on majority judgment. What are its advantages and disadvantages over other voting models?