The problem of dividing a cake fairly is similar in some ways to estate division. Like the cash in an estate, a cake can be divided in any number of ways. Moreover, the participants in a cake division may not agree on the value of a particular slice of cake, just as the heirs to an estate may place different values on a house. However, unlike estate division and legislative apportionment, a cake division problem is strictly continuous and involves no items such as cash and legislative seats that are of indisputable value.

The problem of dividing a continuous medium fairly is not limited to the activities of children. States and countries, for example, must agree on how to share water resources.

Before beginning your exploration of cake division, you may want to review the work that you did in Lesson 2.1. In that lesson, you concluded that the resolution of a cake division problem by an outside authority might not yield a solution that is fair in the eyes of both individuals. You also concluded that the use of a random event such as a coin toss might not result in a division that is fair in the eyes of both parties. For two people to feel that a piece of cake has been divided fairly, each must feel that he or she received at least half the cake.
A good model for the problem of dividing a piece of cake fairly is impossible without a definition of fairness. Therefore, this lesson uses the following definition: A division among $n$ people is called fair if each person feels that he or she receives at least $1/n$th of the cake (or other object).

Lesson 2.1 also showed that an appeal to a measurement scheme such as weighing may not be adequate because an individual’s evaluation of a piece of cake can be based on more than just size. Cake icing, for example, could be more valuable to one person than to another. Along with a definition of fairness, a good model for the cake division problem requires a few realistic assumptions about the participants.

1. Each individual is capable of dividing a portion of the cake into several portions that the individual feels are equal.

2. Each individual is capable of placing a value on any portion of the cake. The total of the values a person places on all portions of a cake is 1, or 100%.

3. The value that each individual places on a portion of the cake can be based on more than just the size of the portion.
The two-person cake division problem is often solved first by having one person cut the cake into two pieces. Then the other person chooses one of them. It is instructive to examine the role of the above definition and assumptions in this model.

The requirement that one person (the cutter) cut the cake into two portions that the person feels are equal is the first assumption. The second assumption ensures that the second person (the chooser) places values that total 1 on the two pieces. However, because of the third assumption, the chooser need not place the same value on both pieces. Even if the chooser feels that his or her piece is more than half the cake, the division is fair because the definition of fairness requires only that each person feel that his or her piece is at least half the cake.

No model for the problem of dividing a cake fairly among three or more people is adequate unless it adheres to the definition of a fair cake division. For example, a fair division among three people must result in each person receiving a piece the person feels is at least one-third of the cake.

In mathematical modeling, often there is more than one way to solve a problem. That is the case with the problem of dividing a cake fairly among three or more people. Sometimes a model can be based on one for a simpler problem. The following model for the three-person problem is based on the two-person model.

Call the three individuals Ava, Bert, and Carlos. The model is described in algorithmic form:

1. Ava cuts the cake into two pieces that she feels are equal.
2. Bert chooses one of the pieces; Ava gets the other.
3. Ava cuts her piece into three pieces that she considers equal; Bert does the same with his.
4. Carlos chooses one of Ava’s three pieces and one of Bert’s three pieces.
To see that this is indeed a solution to the three-person problem, you must be satisfied that the model adheres to the definition of fairness. That is, you must believe that each person receives a portion that she or he feels is at least one-third of the cake.

Consider Ava. In step 1, Ava feels that each piece is one-half of the cake. She therefore feels that the piece she received in step 2 is half the cake. She feels that each piece she cut in step 3 is one-third of half the cake, or one-sixth of the cake. She therefore feels that she receives two-sixths, or one-third, of the cake.

Bert’s case is similar except that he might feel that the portion he chose in step 2 is more than half the cake. Thus, he may feel that each piece he cut in step 3 is more than one-sixth of the cake and that his final share is more than two-sixths, or one-third.

Carlos’ case is different from Ava’s or Bert’s. He might feel that the two pieces Ava cut in step 1 are unequal. He might, for example, consider one piece 0.6 of the cake and the other 0.4. Likewise, he might not feel that the cuts made in step 3 produced equal pieces. He could decide that the piece he valued at 0.6 is divided into pieces he values at 0.3, 0.2, and 0.1. Similarly, he could decide that the piece he values at 0.4 is divided into pieces he values at 0.2, 0.1, and 0.1. However, because he chooses first, he picks the largest piece from each: 0.3 from the piece he values at 0.6 and 0.2 from the piece he values at 0.4. Thus, in this example he feels that the value of the portion he receives is 0.3 + 0.2 = 0.5.

The previous example gives Carlos a portion he values at more than one-third, but it is only one example. To see that Carlos always gets a portion he values at least one-third, use the variable $x$ to represent the value Carlos places on one of the pieces cut by Ava in step 1. His value for the other piece cut by Ava must be $1 - x$. Although Carlos may not feel that the piece he values at $x$ is divided into equal thirds, he chooses the largest of the three pieces and values it as at least one-third of $x$ or $\frac{1}{3}x$. Similarly, he values the piece he chooses from the part he values as $1 - x$ as at least $\frac{1}{3}(1 - x)$. Thus, the total value of his two pieces is at least $\frac{1}{3}x + \frac{1}{3}(1 - x) = \frac{1}{3}x + \frac{1}{3} - \frac{1}{3}x = \frac{1}{3}$.

This model is referred to as the cut-and-choose model.

This lesson’s exercises consider several issues related to cake division and several models for solving problems with three or more participants.
Lesson 2.5 • Fair Division Models: The Continuous Case

Exercises

1. a. In the division among Ava, Bert, and Carlos, who values his or her share as exactly one-third?
   
   b. Who might feel that he or she received more than one-third? Explain.

2. a. Does the division among Ava, Bert, and Carlos result in three portions or three pieces?
   
   b. Does your answer to part a violate the definition of fairness or any of the three assumptions?

3. For each of the four steps of the division among Ava, Bert, and Carlos, state which of the three fairness assumptions is applied.

In Exercises 4 and 5, suppose Carlos feels that Ava’s initial division in step 1 is even, that Ava’s subdivision in step 3 is also even, but that Bert’s subdivision in step 3 is not. (Give your answers as fractions or as decimals rounded to the nearest 0.1.)

4. What value does Carlos place on the piece he takes from Ava? Explain.

5. Although Carlos feels that the piece Bert divided is half the cake, he does not feel that Bert subdivided it equally. He could, for example, place values of 0.3, 0.1, and 0.1 or values of 0.4, 0.06, and 0.04 on the three pieces.

   a. Explain why the largest value Carlos can place on any of Bert’s three subdivisions is 0.5.
   
   b. What is the smallest value Carlos can place on the piece he takes from Bert? Explain.
   
   c. What is the largest total value Carlos can place on the two pieces he takes from Ava and Bert?
   
   d. What is the smallest value Carlos can place on the two pieces he takes from Ava and Bert?

6. Alisha and Balavan are dividing a cake by the cut-and-choose model. After the cake is cut, Alisha thinks that the values of the two pieces are 60% and 40%. Balavan thinks that the values are 50% and 50%. Who did the cutting? Explain.
7. Is the problem of dividing cookies fairly among three children discrete or continuous? Explain.

8. In mathematics, a fundamental principle of counting states: If there are \( m \) ways of performing one task and \( n \) ways of performing another, then there are \( m \times n \) ways of performing both. For example, a tossed coin can fall in two ways, and a rolled die can fall in six ways. Together they can fall in a total of \( 2 \times 6 = 12 \) ways.

a. If two people each have a piece of cake and each person cuts his or her piece into three pieces, show how to use the fundamental counting principle to determine the number of pieces that result.

b. If \( k \) people each have a piece of cake and each cuts his or her piece into \( k + 1 \) pieces, what is an expression for the total number of pieces that result? Show how to use the distributive property to write an equivalent expression.

c. If \( k + 1 \) boxes each contain \( k + 5 \) toothpicks, what are two equivalent expressions for the total number of toothpicks?
d. Two offices are being filled in an election: mayor and governor. If there are three candidates for governor and four for mayor and conventional voting procedures are used, in how many ways can a person vote?

9. Consider the following division of a cake among three people: Arnold, Betty, and Charlie. Arnold cuts the cake into three pieces. Betty chooses one of the pieces, and Charlie chooses either of the remaining two. Arnold gets the third piece.

a. Does Arnold feel he receives at least one-third of the cake? Might he feel he receives more than one-third?

b. Does Betty feel she receives at least one-third of the cake? Might she feel she receives more than one-third?

c. Does Charlie feel he receives at least one-third of the cake? Might he feel he receives more than one-third?

10. Arnold, Betty, and Charlie decide to divide a cake in the following way: Arnold cuts a piece that he considers one-third of the cake. Betty inspects the piece. If she feels it is more than one-third of the cake, she must cut enough from it so that she feels it is one-third of the cake. The removed portion is returned to the cake. Charlie now inspects Betty’s piece and has the option of doing similarly if he thinks that it is more than one-third of the cake. The piece of cake is given to the last person who cuts from it.

Next, one of the remaining two people slices a piece that he or she feels is half of the remaining cake. The other person inspects the piece with the option of removing some of it if he or she feels that it is more than half the remaining cake.

a. Does the person who receives the first piece feel that it is at least one-third of the cake? Could he or she feel it is more than one-third? Explain your answers.
b. Does the person who receives the second piece feel that it is at least one-third of the cake? Could he or she feel it is more than one-third? Explain your answers.

c. Does the person who receives the third piece feel that it is at least one-third of the cake? Could he or she feel it is more than one-third? Explain your answers.

This model is called the **inspection model**.

11. The definition of a fair cake division used in this lesson does not state that each person must feel his or her portion of the cake is at least as large as the portion that each of the other participants receives.

   a. Construct an example for the three-person cut-and-choose division among Ava, Bert, and Carlos to show that Carlos may feel that the piece received by one of the other two is better than his.

   b. Could the definition of fair cake division result in jealously? Explain.

**Computer/Calculator Exploration**

12. Use the moving knife program that accompanies this book to divide a cake of any shape amongst yourself and two other people by means of the **moving knife model**.

   a. Explain why each of the people in your group feels that he or she received at least one-third of the cake.

   b. Might any of the people feel jealous of the share received by another? Explain.

13. A model for dividing a cake fairly can be considered “optimal” if it fulfills some criterion. For example, a model might be called optimal if it results in exactly the same number of pieces as participants. By this criterion, is the cut-and-choose model, the inspection model, or the moving knife model optimal? Explain.

14. Could the cut-and-choose, the inspection, or the moving knife model be extended to divide a cake among four people? Explain how to do this.
Chimps Possess a Sense of 'Fairness'

ABC News
January 15, 2013

Sharing and caring
Chimpanzees, in a test of their willingness to share with other chimps, displayed a surprising sense of fairness, scientists say, debunking the idea that only humans boast that quality.

The researchers say they are the first to show chimpanzees have a sense of fairness and that this may have evolved over time to aid their survival.

The experiments were carried out by scientists at the Yerkes National Primate Research Center at Emory University near Atlanta, Georgia, working with colleagues at Georgia State University.

They taught the chimps to play the game 'Ultimatum' in which "one individual needs to propose a reward division to another individual, and then have that individual accept the proposition before both can obtain the rewards," says researcher Frans de Waal.

Researchers ran the test separately on six adult chimpanzees and 20 human children between the ages of two and seven.

The game was simple: One individual chose between two differently colored tokens that, with his or her partner's cooperation, could be exchanged for rewards – a snack item for the chimps, or stickers for the children.

A token of one color offered equal rewards to both players, while the other gave the individual making the choice a much larger share of the spoils.

The results showed no differences between human and chimp behavior: If the partner's cooperation was required, the chimpanzees and children split the rewards equally, researchers write.

The scientists note that from an evolutionary standpoint, chimpanzees are highly cooperative in the wild and would need to be sensitive to reward distributions for survival.