In Lesson 3.4, you found that it was possible to use an initial population distribution along with birth and survival rates to predict population numbers at future times. As you explored your model, you found that you could look 2, 3, or even 4 cycles into the future. However, the arithmetic soon became cumbersome. What do wildlife managers and urban planners do if they want to look 10, 20, or even more cycles into the future?

In Lesson 3.4 (Exercises 2 and 3), you began to get a glimpse of the model that Leslie proposed. The use of matrices seems to hold the key. And with the aid of computer software or a calculator, looking ahead many cycles is not difficult. In fact, some very fascinating results are produced.

Return to the original rat model. If you multiply the original population distribution \( P_0 \) times a matrix that we will call \( L \), you can calculate the population distribution at the end of cycle 1 \( P_1 \).

\[
P_0L = \begin{bmatrix}
15 & 9 & 13 & 5 & 0 & 0 \\
0.6 & 0 & 0 & 0 & 0 & 0 \\
0.3 & 0 & 0.9 & 0 & 0 & 0 \\
0.8 & 0 & 0 & 0.9 & 0 & 0 \\
0.7 & 0 & 0 & 0 & 0.8 & 0 \\
0.4 & 0 & 0 & 0 & 0 & 0.6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
15(0) + 9(0.3) + 13(0.8) + 5(0.7) + 0(0.4) + 0(0) \\
15(0.6) + 9(0.9) + 13(0.9) + 5(0.8) + 0(0.6)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
16.6 & 9.0 & 8.1 & 11.7 & 4.0 & 0
\end{bmatrix}
\]

\( = P_1 \)
The matrix $L$ is called the **Leslie matrix**. This matrix is formed by augmenting or joining the column matrix containing the birth rates of each age group and a series of column matrices that contain the survival rates. Notice that the survival-rate columns contain the survival rates as one entry and zeros everywhere else. The survival rates (of which there is one less than the actual number of survival rates since no animal survives beyond the 15–18 age group) lie along the **super diagonal** that is immediately above the main diagonal of the matrix.

When the matrix $L$ is multiplied by a population distribution $P_k$, a new population distribution $P_{k+1}$ results. To find population distributions at the end of other cycles, the process can be continued.

$$
P_1 = P_0 L
$$

$$
P_2 = P_1 L = (P_0 L) L = P_0 (LL) = P_0 L^2
$$

In general, $P_k = P_0 L^k$.

### Example

Use the formula $P_k = P_0 L^k$ to find the population distribution for the rats after 24 months (8 cycles) and the total population of the rats.

#### Solution:

The population distribution after 8 cycles is

$$
P^8 = P_0 L^8 = \begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.9 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0.9 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^8
$$

$$
= \begin{bmatrix} 21.03 & 12.28 & 10.90 & 9.46 & 7.01 & 4.27 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Total population = 21.03 + 12.28 + 10.90 + 9.46 + 7.01 + 4.27 = 64.95, or approximately 65 rats.

**Technology Note**

You can perform the calculation $P_0 L^8$ on a calculator with matrix features.

```
[a] [b]^8
[[21.026 12.283...]]
```
Exercises

Note: For the following exercises, you need to have access to either a graphing calculator or computer software that performs matrix operations.

1. Use the original population distribution, \([15 \ 9 \ 13 \ 5 \ 0 \ 0]\), and the Leslie matrix from the *Rattus norvegicus* example to find the following.
   a. The population distribution after 15 months (5 cycles)
   b. The total population after 15 months (Hint: Multiply \(P_0 L^5\) times a column matrix consisting of six 1s.)
   c. The population distribution and the total population after 21 months

2. Suppose the *Rattus norvegicus* start dying off from overcrowding when the total female population for a colony reaches 250. Find how long it will take for this to happen when the initial population is
   a. \([18 \ 9 \ 7 \ 0 \ 0 \ 0]\).
   b. \([35 \ 0 \ 0 \ 0 \ 0 \ 0]\).
   c. \([5 \ 5 \ 5 \ 5 \ 5 \ 5]\).
   d. \([25 \ 15 \ 10 \ 11 \ 7 \ 13]\).

3. a. Complete the table for the given cycles of *Rattus norvegicus* using the original population distribution of \([15 \ 9 \ 13 \ 5 \ 0 \ 0]\).

<table>
<thead>
<tr>
<th>Cycle</th>
<th>Total Population</th>
<th>Growth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>42</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>49.4</td>
<td>(\frac{49.4 - 42}{42} = 0.176 = 17.6%)</td>
</tr>
<tr>
<td>2</td>
<td>56.08</td>
<td>(\frac{56.08 - 49.4}{49.4} = 0.135 = 13.5%)</td>
</tr>
<tr>
<td>3</td>
<td>57.40</td>
<td>(\frac{57.40 - 56.08}{56.08} = 0.024 = 2.4%)</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
b. What do you observe about the growth rates?

c. Calculate the total populations for \(P_{25}\), \(P_{26}\), and \(P_{27}\). What is the growth rate between these successive years? Hint: To find the growth rate from \(P_{25}\) to \(P_{26}\), subtract the total population for \(P_{25}\) from the total population for \(P_{26}\) and divide the result by the total population for \(P_{25}\).

4. One characteristic of the Leslie model is that growth does stabilize at a rate called the \textit{long-term growth rate} of the population. As you observed in Exercise 3, the growth rate of \textit{Rattus norvegicus} converges to about 3.04%. This means that for a large enough \(k\), the total population in cycle \(k\) will equal about 1.0304 times the total population in the previous cycle.

a. Find the long-term growth rate of the total population for each of the initial population distributions in Exercise 2.

b. How does the initial population distribution seem to affect the long-term growth rate?

5. Again, consider the deer species from Lesson 3.4. The birth and survival rates follow.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Birth Rate</th>
<th>Survival Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–2</td>
<td>0</td>
<td>0.6</td>
</tr>
<tr>
<td>2–4</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>4–6</td>
<td>1.7</td>
<td>0.9</td>
</tr>
<tr>
<td>6–8</td>
<td>1.7</td>
<td>0.9</td>
</tr>
<tr>
<td>8–10</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>10–12</td>
<td>0.4</td>
<td>0</td>
</tr>
</tbody>
</table>

a. Construct the Leslie matrix for this animal.

b. Given that \(P_0 = [50  30  24  24  12  8]\), find the long-term growth rate.

c. Suppose the natural range for this animal can sustain a herd that contains a maximum of 1,250 females. How long before this herd size is reached?

d. Once the long-term growth rate of the deer population is reached, how might the population of the herd be kept constant?
In his study of the application of matrices to population growth, P. H. Leslie was particularly interested in the special case in which the birth rate matrix has only one nonzero element. The following example falls into this special case.

6. Suppose there is a certain kind of bug that lives at most 3 weeks and reproduces only in the third week of life. Fifty percent of the bugs born in one week survive into the second week, and 70% of the bugs that survive into their second week also survive into their third week. On the average, six new bugs are produced for each bug that survives into its third week. A group of five 3-week-old female bugs decide to make their home in a storage box in your basement.

a. Construct the Leslie matrix for this bug.
b. What is \( P_0 \)?
c. How long will it be before there are at least 1,000 female bugs living in your basement?

7. Exercise 6 is an example of a population that grows in waves. Will the population growth for this population stabilize in any way over the long run? To explore this question, make a table of the population distributions \( P_{22} \) through \( P_{30} \).

a. Examine the population change from one cycle to the next. Can you find a pattern in the population growth?
b. Examine the population change from \( P_{22} \) to \( P_{25} \), \( P_{23} \) to \( P_{26} \), \( P_{24} \) to \( P_{27} \), \( P_{25} \) to \( P_{28} \), \( P_{26} \) to \( P_{29} \), and \( P_{27} \) to \( P_{30} \). Are you surprised at the results? Why?

8. a. Change the initial population in Exercise 6 to \( P_0 = [4 \ 4 \ 4] \) and repeat the instructions in Exercise 7 looking at the total population growth for each cycle.
b. Examine the changes in successive age groups from \( P_{22} \) to \( P_{25} \), \( P_{23} \) to \( P_{26} \), \( P_{24} \) to \( P_{27} \), \( P_{25} \) to \( P_{28} \), \( P_{26} \) to \( P_{29} \), and \( P_{27} \) to \( P_{30} \). Make a conjecture based on your results.

9. Using mathematical induction, prove that \( P_k = P_0 L^k \) for any original population \( P_0 \) and Leslie matrix \( L \), where \( k \) is a natural number.

**Modeling Projects**

10. Search the Web for applications of the Leslie matrix model in managing wildlife or domestic herds.