Lesson 4.5

Hamiltonian Circuits and Paths

Since its inception, graph theory has been closely tied to applications. In Lesson 4.4, you investigated situations in which you needed to traverse each edge of a graph. In this lesson, you will explore situations that can be modeled with graphs in which each vertex must be visited.

Explore This

Suppose once again that you are a city inspector, but instead of inspecting all of the streets in an efficient manner, you must inspect the fire hydrants that are located at each of the street intersections. This implies that you are searching for an optimal route that begins at the garage *G*, visits each intersection exactly once, and returns to the garage (see Figure 4.11).

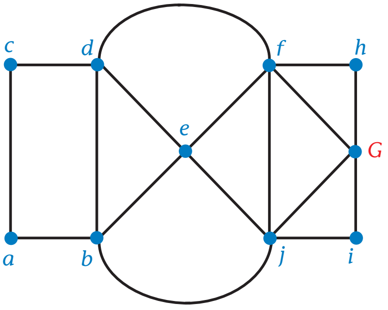


Figure 4.11. Street network.

One path that meets these criteria is *G, h, f, d, c, a, b, e, j, i, G*. Notice that it is not necessary to traverse every edge of the graph when visiting each vertex exactly once.

Mathematician of Note

Sir William Rowan Hamilton (1805–1865)

Hamilton, a leading nineteenth-century Irish mathematician, was appointed Astronomer Royal of Ireland at the age of 22 and knighted at 30. Shortly before his death, Hamilton received word that he had been elected the first foreign member of the National Academy of Sciences of the USA. Hamiltonian Mechanics is still used today to determine orbital trajectories of satellites.

In 1856, Sir William Rowan Hamilton used his mathematical knowledge to create a game called the Icosian game. The game consisted of a graph in which the vertices represented major cities in Europe. The object of the game was to find a path that visited each of the 20 vertices exactly once. In honor of Hamilton and his game, a path that uses each vertex of a graph exactly once is known as a **Hamiltonian path**. If the path ends at the starting vertex, it is called a **Hamiltonian circuit**.

Try to find a Hamiltonian circuit for each of the graphs in Figure 4.12.

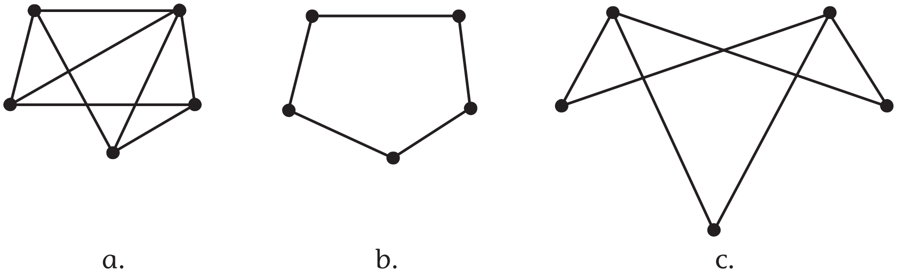


Figure 4.12. Graphs with possible Hamiltonian circuits.

Mathematicians continue to be intrigued with this type of problem because a simple test for determining whether a graph has a Hamiltonian circuit has not been found. It now appears that a general solution that applies to all graphs may be impossible. Fortunately, several theorems have been proved that help guarantee the existence of Hamiltonian circuits for certain kinds of graphs. The following is one such theorem.

If a connected graph has n vertices, where $n > 2$ and each vertex has degree of at least $n/2$, then the graph has a Hamiltonian circuit.

To apply the Hamiltonian circuit theorem to Figure 4.12a, check the degree of each vertex. Since each of the five vertices of the graph has degree of at least 5/2, the theorem guarantees that the graph has a Hamiltonian circuit. Unfortunately, it does not tell you how to find it.

The theorem has another downside as well. If a graph has a vertex with degree less than $n/2$, then the theorem simply does not apply to that graph. It may or it may not have a Hamiltonian circuit. Each of the graphs in parts b and c of Figure 4.12 has some vertices of degree less than 5/2, so no conclusions can be drawn. By inspection, Figure 4.12b has a Hamiltonian circuit, but Figure 4.12c does not.

Tournaments

As with Euler circuits, often it is useful for the edges of the graph to have direction. Consider a competition in which each player must play every other player. By using directed edges, it is possible to indicate winners and losers. To illustrate this, draw a complete graph in which the vertices represent the players, and a directed edge from vertex *A* to vertex *B* indicates that player *A* defeats player *B*. This type of graph is known as a tournament. A **tournament** is a digraph that results from giving directions to the edges of a complete graph. Figure 4.13 shows a tournament in which *A* beats *B*, *C* beats *B*, and *A* beats *C*.

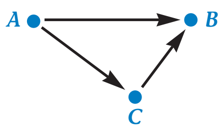


Figure 4.13. Tournament with three vertices.

One interesting property of such a digraph is that every tournament contains at least one Hamiltonian path. If there is exactly one such path, it can be used to rank the teams in order, from winner to loser.

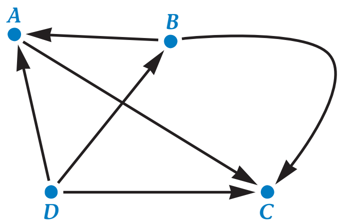
Example

Suppose four teams play in the school soccer round-robin tournament. The results of the competition follow:

Game	AB	AC	AD	BC	BD	CD
Winner	B	A	D	B	D	D

Draw a digraph to represent the tournament. Find a Hamiltonian path and use it to rank the participants from winner to loser.

Solution:

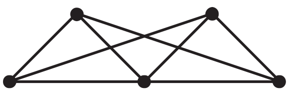


To determine a ranking, remember that a tournament results from a complete graph when direction is given to the edges. In this case, there is only one Hamiltonian path for the graph: *D, B, A, C*. Therefore, *D* finishes first, *B* is second, *A* is third, and *C* finishes fourth.

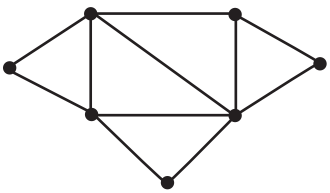
Exercises

1. Apply the theorem from page 202 to the following graphs. According to the theorem, which of the graphs have Hamiltonian circuits? Explain your reasoning.

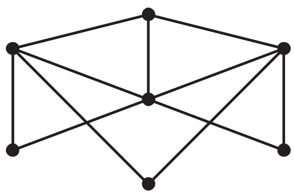
a.



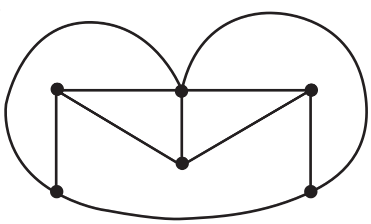
b.



c.



d.

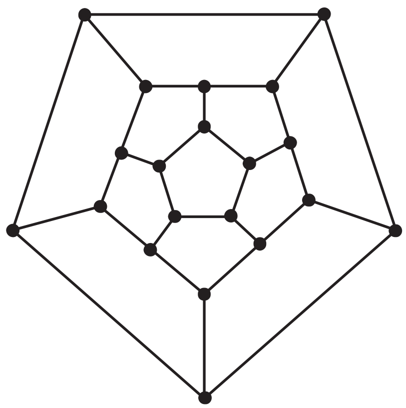


2. Give two examples of situations that could be modeled by a graph in which finding a Hamiltonian path or circuit would be of benefit.

3. a. Construct a graph that has both an Euler and a Hamiltonian circuit.

   b. Construct a graph that has neither an Euler nor a Hamiltonian circuit.

4. Hamilton's Icosian game was played on a wooden regular dodecahedron (a solid figure with 12 sides). Here is a planar representation of the game.

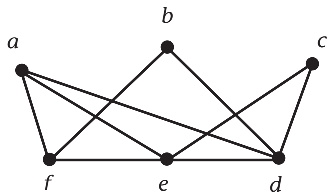


   a. Copy the graph onto your paper and find a Hamiltonian circuit for the graph.

   b. Is there only one Hamiltonian circuit for the graph?

   c. Can the circuit begin at any of the vertices or only some of them?

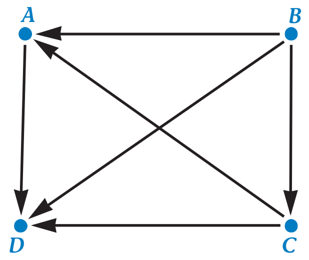
5. Find a Hamiltonian circuit for the following graph.



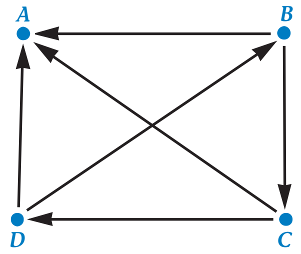
6. Draw a tournament with five players, in which player A defeats everyone, B defeats everyone but A, C is defeated by everyone, and D defeats E.

7. Find all the directed Hamiltonian paths for each of the following tournaments:

a.



b.



8. Draw a tournament with three vertices in which:
   a. One player wins all the games he or she plays.
   b. Each player wins exactly one game.
   c. Two players lose all of the games they play.

9. Draw a tournament with five vertices in which there is a three-way tie for first place.

10. When ties exist in a ranking for a tournament (e.g., more than one first place winner), there is more than one Hamiltonian path for the graph. Explain why this is so.

11. a. Write an algorithm that uses the outdegree of the vertices to find the Hamiltonian path for a tournament that has exactly one Hamiltonian path.
    b. Explain the difficulties that arise with your algorithm when the tournament has more than one Hamiltonian path.

12. Complete the following table for a tournament.

Number of Vertices	Sum of the Outdegrees of the Vertices
1	0
2	1
3	3
4	
5	
6	

Write a recurrence relation that expresses the relationships between S_n, the sum of the outdegrees for a tournament with n vertices, and S_{n-1}.

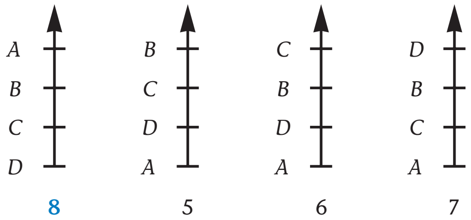
13. In a tournament a **transmitter** is a vertex with a positive outdegree and a zero indegree. A **receiver** is a vertex with a positive indegree and a zero outdegree. Explain why a tournament can have at most one transmitter and at most one receiver.

14. Use mathematical induction on the number of vertices to prove that every tournament has a Hamiltonian path.

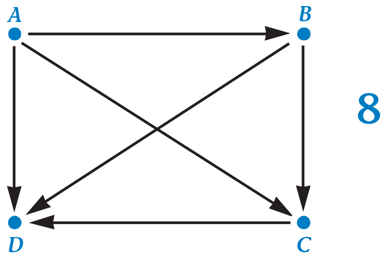
a. Begin the mathematical induction process by showing that every tournament with one vertex has a Hamiltonian path.

b. Assume that a tournament of k vertices has a Hamiltonian path and use this assumption to prove that a tournament of $k + 1$ vertices has a Hamiltonian path.

15. Consider the set of preference schedules from Lesson 1.3:



The first preference schedule could be represented by the following tournament.



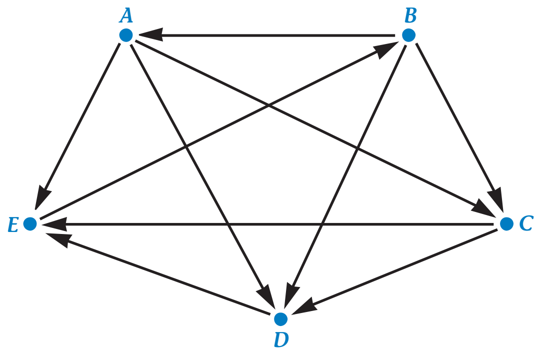
a. Construct tournaments for each of the three other preference schedules.

b. Construct a cumulative preference tournament that would show the overall results of the four individual preference schedules.

c. Is there a Condorcet winner in the election? (Recall from Lesson 1.3 that a Condorcet winner is one who is able to defeat each of the other choices in a one-on-one contest.)

d. Find a Hamiltonian path for the cumulative tournament. What does this path indicate?

16. a. Construct an adjacency matrix for the following digraph, and call the matrix M.



By summing the rows of M, you can see that a tie exists between A and B, each has three wins.

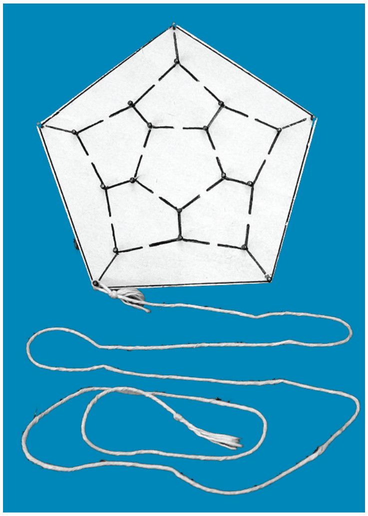
b. Square M. Notice that this new matrix M^2 gives the number of paths of length 2 between vertices. For example, the 3 in entry M_{25} indicates that 3 paths of length 2 exist between B and E. These paths are B, C, E; B, D, E; and B, A, E. In the case of a tournament, this means that B has three 2-stage wins, or **dominances**, over E.

c. Add M and M^2. Use the sum to determine the total number of ways that A and B can dominate in one and two stages. Who might now be considered the winner?

d. What would M^3 indicate? Find M^3 and see whether you are correct.

Project

17. Design and build a Planar Icosian Game by enlarging the graph in Exercise 4 and copying it onto a piece of plywood or heavy cardboard. Use tacks for the vertices. The game is then played by tying a piece of string (approximately 12 inches or longer) on one of the tacks (vertices) and attempting to wind the string from tack to tack (following the lines on the graph) until all of the tacks are touched and the player is back to the initial tack (in other words, until a Hamiltonian circuit is found). Try the game with younger children and adults. Who seems to find a Hamiltonian circuit the quickest?



Student-Made Icosian Game