Elaborate communication networks span the country and most of the earth. The ability to transmit information quickly and easily through these networks affects many aspects of our lives—the way we work, the way we learn, and the way in which we are entertained.

- How are communication networks that link several locations together constructed at the least possible cost?
- How is the most efficient route between two locations in a network found?
- Can the models used to find the best route between points in a communication network also be used to plan the best route for an automobile or plane trip?

The mathematics of graph theory plays an important role in solving these and many other problems that are important in our ever-changing world.
In Lesson 4.6, problems involving conflict were solved by modeling them with graphs and then coloring the graphs. The four-color theorem states that any map that can be drawn on the surface of a sphere can be colored with four colors or fewer. If this is true, then why does it take more than four colors to color some graphs?

**Explore This**

Try to redraw the graphs in Figures 5.1 and 5.2 so that their edges intersect only at the vertices. Try to think of the edges of the graph as rubber bands.

![Figure 5.1. $K_4$ graph.](image1)

![Figure 5.2. $K_5$ graph.](image2)

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![Figure 5.1. $K_4$ graph.](image1)

![Figure 5.2. $K_5$ graph.](image2)
It is relatively easy to redraw Figure 5.1 so that the edges do not cross (see Figure 5.3). But no matter how hard you try, at least two edges of Figure 5.2 will always intersect (see Figure 5.4). A graph that can be drawn so that no two edges intersect except at a vertex is called a **planar graph**. Figure 5.1 shows a planar graph. Figure 5.2 shows a graph that is not planar.

If regions of a map are represented by vertices of a graph and edges are drawn between vertices when boundaries exist between regions, the resulting graph is planar. In other words, when a map in a plane or on a sphere is modeled by a graph, the resulting graph is always planar.

Hence, the four-color theorem can be stated in a different way:

**Every planar graph has a chromatic number that is less than or equal to four.**

The question asked earlier about why some graphs require more than four colors can now be answered. Planarity is the key. If a graph is not planar, we do not know how many colors it will take to color it.

One type of graph that is not planar, a $K_5$, is shown in Figure 5.2. Another nonplanar graph about which many problems have been written is shown in Figure 5.5. Try to redraw it without the edges crossing. Once again you will discover that this is not possible.
Bipartite Graphs

The graph in Figure 5.5 has interesting characteristics other than the fact that it is not planar. It is one example of a group of graphs known as bipartite graphs. A graph is bipartite if its vertices can be divided into two distinct sets so that each edge of the graph has one vertex in each set.

A bipartite graph is said to be complete if it contains all possible edges between the pairs of vertices in the two distinct sets. Complete bipartite graphs can be denoted by $K_{m,n}$, where $m$ and $n$ are the number of vertices in the two distinct sets. Hence, Figure 5.5 is a $K_{3,3}$ graph.

Figure 5.6 is an example of a complete bipartite graph $K_{3,2}$, since its vertices can be separated into two distinct sets {A, B, C} and {X, Y}, every edge has one vertex in each set, and all possible edges from one set to the other are drawn.

![Figure 5.6. $K_{3,2}$ graph.](image)

One way to determine whether a given graph is planar is to try to redraw the graph without edges crossing. For a very large graph, this could prove to be both difficult and time-consuming. In 1930 Kazimierz Kuratowski, a Polish mathematician, provided a partial resolution to this problem of determining the planarity of a graph. He proved that if a graph has a $K_5$ or $K_{3,3}$ subgraph, it is not planar. A graph $G'$ is said to be a subgraph of graph $G$ if all of the vertices and edges of $G'$ are contained in $G$.

In addition to proving that graphs with $K_5$ or $K_{3,3}$ subgraphs are not planar, Kuratowski proved that the inverse of his theorem is not true. That is, the lack of a $K_5$ or $K_{3,3}$ subgraph does not guarantee that the graph is planar (see Exercises 23 and 24 on pages 239 and 240).

**Point of Interest**

In practice, approximately 99% of all nonplanar graphs of modest size can be shown to be nonplanar because of a $K_{3,3}$ subgraph rather than a $K_5$ subgraph.
Example

Determine whether the following graph is planar or nonplanar.

Solution:

On close inspection of the graph and vertices, A, B, C, D, E, and F, a $K_{3,3}$ subgraph can be found. Therefore, according to Kuratowski’s theorem, the graph is nonplanar.

Exercises

In Exercises 1 through 4, decide whether the graph is planar or nonplanar. If the graph is planar, redraw it without edge crossings.
5. The following graph is planar. Draw it without edge crossings.

6. By looking at the graph in Exercise 5, how can you tell that it does not contain a $K_5$ subgraph?

7. Devise a systematic method of searching a graph for a $K_5$ subgraph. Describe your method in a short paragraph and try it on the following graph. Does the graph contain a $K_5$ subgraph?

8. The complement of a graph $G$ is customarily denoted by $\overline{G}$. The complement $\overline{G}$ has the same vertices as $G$, but its edges are those not in $G$. The edges of $G$ and $\overline{G}$ along with vertices from either set would make a complete graph. Draw the complement of the following graph.
9. Every planar graph with nine vertices has a nonplanar complement. Verify this statement for one case by drawing a planar graph with nine vertices and then drawing its complement. For your case, is the complement nonplanar?

10. The concept of planarity is extremely important to printing circuit boards for the electronics industry. Explain why.

11. Construct the following bipartite graphs.
   a. \( K_{2,3} \)  
   b. \( K_{2,4} \)

12. For each of the following bipartite graphs, list the two distinct sets into which the vertices can be divided.
   a. 
   b. 
   c. 
   d. 

13. State whether the following graphs are bipartite. Explain why or why not.
   a. 
   b. 
   c. 

14. Devise a method of telling whether a graph is bipartite. Write a short algorithm for your method.

15. How many edges are in a $K_{2,3}$ graph? A $K_{4,3}$ graph? A $K_{m,n}$ graph?

16. When does a bipartite graph $K_{m,n}$ have an Euler circuit?

17. What is the chromatic number of a $K_{m,n}$ graph?

18. At Ms. Johnson’s party, six men and five women walk into the dining room. If each man shakes hands with each woman, how many handshakes will occur? Represent this situation with a graph. What kind of a graph is it?

19. Describe a situation that can be represented by a bipartite graph that is not complete.

20. The following puzzle is often referred to as the *Wells and Houses problem* or the *Utilities problem*.

Three houses and three wells are built on a piece of land in an arid country. Because it seldom rains, the wells often run dry, and so each house must have access to each well. Unfortunately, the occupants of the three houses dislike one another and want to construct paths to the wells so that no two paths cross.

Draw a graph to model this problem. Is it possible to satisfy the wishes of the feuding families? Explain why or why not.
21. The preceding graph is a subgraph of which following graph(s)? Explain how you know.

   a.  
   b.  
   c.  
   d.  

22. Kuratowski proved the following conditional statement.

   If a graph contains a $K_5$ or a $K_{3,3}$ subgraph, then it is not planar.

   He also proved that the inverse of the statement is false:

   If a graph does not contain a $K_5$ or a $K_{3,3}$ subgraph, then it is planar.

   State the converse of the conditional statement. Do you think that it is true or false?

23. Is the following graph planar? Does it contain a $K_5$ or a $K_{3,3}$ subgraph?
24. For many years mathematicians thought that all nonplanar graphs had either a $K_5$ or a $K_{3,3}$ subgraph. Then nonplanar graphs such as the one in Exercise 23 were found. The graph in Exercise 23 is said to be an extension of a $K_5$ because it was formed by adding a vertex or vertices to the edges of the $K_5$ graph. Extensions of $K_5$ and $K_{3,3}$ graphs are nonplanar. This discovery shows that the converse of Kuratowski's theorem is false (Exercise 22).

Redraw the graph in Exercise 23 to show that it is an extension of a $K_5$. 