Limits, Continuity, and Differentiability

Continuity
A function is continuous on an interval if it is continuous at every point of the interval. Intuitively, a function is continuous if its graph can be drawn without ever needing to pick up the pencil. This means that the graph of $y = f(x)$ has no “holes”, no “jumps” and no vertical asymptotes at $x = a$. When answering free response questions on the AP exam, the formal definition of continuity is required. To earn all of the points on the free response question scoring rubric, all three of the following criteria need to be met, with work shown:

A function is **continuous** at a point $x = a$ if and only if:
1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$ (i.e., the limit equals the function value)

Continuity and Differentiability
Differentiability implies continuity (but not necessarily vice versa) If a function is differentiable at a point (at every point on an interval), then it is continuous at that point (on that interval). The converse is not always true: continuous functions may not be differentiable. It is possible for a function to be continuous at a specific value for $a$ but not differentiable there.

Example (graph with a sharp turn): Consider the function $f(x) = |x - 3|:

The graph of $f(x) = |x - 3|$ is continuous at $x = 3$ but the one-sided limits are not equal:

$$\lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3} = \frac{|x - 3| - 0}{x - 3} = -1 \quad \text{and} \quad \lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} = \frac{|x - 3| - 0}{x - 3} = 1$$

so $f$ is not differentiable at $x = 3$ and the graph of $f$ does not have a tangent line at the point $(3, 0)$.
Example (graph with a vertical tangent line) Consider the function $f(x) = 2x^{\frac{1}{3}}$:

The function $f(x) = 2x^{\frac{1}{3}}$ is continuous at $x = 0$ but because the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{2x^{\frac{1}{3}} - 0}{x} = \lim_{x \to 0} \frac{2}{3x^{\frac{2}{3}}} = \infty$$

is infinite, it can be concluded that the tangent line is vertical at $x = 0$, therefore, $f$ is not differentiable at $x = 0$.

Quick Check for Understanding:
1. Sketch a function with the property that $f(a)$ exists but $\lim_{x \to a} f(x)$ does not exist.

2. Sketch a function with the property that $\lim_{x \to a} f(x)$ exists but $f(a)$ does not exist.

3. Sketch a function with the property that $f(a)$ exists and $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$. 

Students should be able to:

- Determine limits from a graph
- Know the relationship between limits and asymptotes (i.e., limits that become infinite at a finite value or finite limits at infinity)
- Compute limits algebraically
- Discuss continuity algebraically and graphically and know its relation to limit.
- Discuss differentiability algebraically and graphically and know its relation to limits and continuity
- Recognize the limit definition of derivative and be able to identify the function involved and the point at which the derivative is evaluated. For example, since
  \[
  f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},
  \]
  recognize that \( \lim_{h \to 0} \frac{\cos(\pi + h) - \cos(\pi)}{h} \) is simply the derivative of \( \cos(x) \) at \( x = \pi \).
- L’Hôpital’s Rule (BC only)
- Limits associated with logistic equations (BC only)
Multiple Choice

1. (calculator not allowed) (1985AB5)
\[ \lim_{n \to \infty} \frac{4n^2}{n^2 + 10000n} \]
   (A) 0
   (B) \frac{1}{2500}
   (C) 1
   (D) 4
   (E) nonexistent

2. (calculator not allowed)
\[ \lim_{x \to \infty} \frac{(2x-1)(3-x)}{(x-1)(x+3)} \]
   (A) -3
   (B) -2
   (C) 2
   (D) 3
   (E) nonexistent

3. (calculator not allowed)
What is
\[ \lim_{h \to 0} \frac{8 \left( \frac{1}{2} + h \right)^8 - 8 \left( \frac{1}{2} \right)^8}{h} \]?
   (A) 0
   (B) \frac{1}{2}
   (C) 1
   (D) The limit does not exist.
   (E) It cannot be determined from the information given.

4. (calculator not allowed)
\[ \lim_{x \to \frac{\pi}{4}} \sin \left( x - \frac{\pi}{4} \right) \]
   (A) 0
   (B) \frac{1}{\sqrt{2}}
   (C) \frac{\pi}{4}
   (D) 1
   (E) nonexistent
5. (calculator not allowed)

The \( \lim_{h \to 0} \frac{\tan(3(x+h)) - \tan(3x)}{h} \) is

(A) 0
(B) \( 3\sec^2(3x) \)
(C) \( \sec^2(3x) \)
(D) \( 3\cot(3x) \)
(E) nonexistent

6. (calculator not allowed)

If \( f(x) = 2x^2 + 1, \) then \( \lim_{x \to 0} \frac{f(x) - f(0)}{x^2} \) is

(A) 0
(B) 1
(C) 2
(D) 4
(E) nonexistent

7. (calculator not allowed)

If \( f'(x) = \cos x \) and \( g'(x) = 1 \) for all \( x, \) and if \( f(0) = g(0) = 0, \) then \( \lim_{x \to 0} \frac{f(x)}{g(x)} \) is

(A) \( \frac{\pi}{2} \)
(B) 1
(C) 0
(D) -1
(E) nonexistent
8. (calculator not allowed)
\[ \lim_{\theta \to 0} \frac{1 - \cos \theta}{2\sin^2 \theta} \]
is

(A) 0

(B) \( \frac{1}{8} \)

(C) \( \frac{1}{4} \)

(D) 1

(E) nonexistent

9. (calculator not allowed)
If \( \lim_{x \to a} f(x) = L \) where \( L \) is a real number, which of the following must be true?

(A) \( f'(a) \) exists.

(B) \( f(x) \) is continuous at \( x = a \).

(C) \( f(x) \) is defined at \( x = a \).

(D) \( f(a) = L \)

(E) None of the above

10. (calculator not allowed)
For \( x \geq 0 \), the horizontal line \( y = 2 \) is an asymptote for the graph of the function \( f \). Which of the following statements must be true?

(A) \( f(0) = 2 \)

(B) \( f(x) \neq 2 \) for all \( x \geq 0 \)

(C) \( f(2) \) is undefined.

(D) \( \lim_{x \to 2} f(x) = \infty \)

(E) \( \lim_{x \to \infty} f(x) = 2 \)
11. (calculator not allowed)
If the graph of \( y = \frac{ax + b}{x + c} \) has a horizontal asymptote at \( y = 2 \) and a vertical asymptote at \( x = -3 \), then \( a + c = \)

(A) \(-5\)
(B) \(-1\)
(C) 0
(D) 1
(E) 5

12. (calculator not allowed)
At \( x = 3 \), the function given by \( f(x) = \begin{cases} 
  x^2, & x < 3 \\
  6x - 9, & x \geq 3
\end{cases} \) is

(A) undefined.
(B) continuous but not differentiable.
(C) differentiable but not continuous.
(D) neither continuous nor differentiable.
(E) both continuous and differentiable.

13. (calculator not allowed)

\[ f(x) = \begin{cases} 
  x + 2, & \text{if } x \leq 3 \\
  4x - 7, & \text{if } x > 3
\end{cases} \]

Let \( f \) be the function given above. Which of the following statements are true about \( f \)?

I. \( \lim_{x \to 3} f(x) \) exists.
II. \( f \) is continuous at \( x = 3 \).
III. \( f \) is differentiable at \( x = 3 \).

(A) None
(B) I only
(C) II only
(D) I and II only
(E) I, II and III
14. (calculator not allowed)

\[ f(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2}, & x \neq 2 \\
1, & x = 2
\end{cases} \]

Let \( f \) be the function defined above. Which of the following statements about \( f \) are true?

I. \( f \) has a limit at \( x = 2 \).
II. \( f \) is continuous at \( x = 2 \).
III. \( f \) is differentiable at \( x = 2 \).

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I, II, and III

15. (calculator not allowed)

Let \( f \) be defined by the following.

\[ f(x) = \begin{cases} 
\sin x, & x < 0 \\
x^2, & 0 \leq x < 1 \\
2 - x, & 1 \leq x \leq 2 \\
x - 3, & x \geq 2
\end{cases} \]

For what values of \( x \) is \( f \) NOT continuous?

(A) 0 only
(B) 1 only
(C) 2 only
(D) 0 and 2 only
(E) 0, 1, and 2

16. (calculator not allowed)

If \( f(x) = 2 + |x - 3| \) for all \( x \), then the value of the derivative \( f'(x) \) at \( x = 3 \) is

(A) \(-1\)
(B) 0
(C) 1
(D) 2
(E) nonexistent
17. (calculator not allowed)

If the function \( f \) is continuous for all real numbers and if \( f(x) = \frac{x^2 - 4}{x + 2} \) when \( x \neq -2 \),
then \( f(-2) = \)

(A) \(-4\)
(B) \(-2\)
(C) \(-1\)
(D) \(0\)
(E) \(2\)

18. (calculator not allowed)

The graph of the function \( f \) is shown in the figure above. Which of the following statements about \( f \) is true?

(A) \( \lim_{x \to a} f(x) = \lim_{x \to b} f(x) \)
(B) \( \lim_{x \to a} f(x) = 2 \)
(C) \( \lim_{x \to b} f(x) = 2 \)
(D) \( \lim_{x \to b} f(x) = 1 \)
(E) \( \lim_{x \to a} f(x) = \) does not exist.
19. (calculator not allowed)

The graph of a function \( f \) is shown above. At which value of \( x \) is \( f \) continuous, but not differentiable?

(A) \( a \)
(B) \( b \)
(C) \( c \)
(D) \( d \)
(E) \( e \)

20. (calculator allowed)

For which of the following does \( \lim_{{x \to 4}} f(x) \) exist?

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I and III only
21. (calculator allowed)

The figure above shows the graph of a function $f$ with domain $0 \leq x \leq 4$. Which of the following statements are true?

I. $\lim_{{x \to 2^-}} f(x)$ exists
II. $\lim_{{x \to 2^+}} f(x)$ exists
III. $\lim_{{x \to 2}} f(x)$ exists

(A) I only     (B) II only     (C) I and II only     (D) I and III only     (E) I, II, and III

22. (calculator allowed)

The graph of the function $f$ is shown in the figure above. The value of $\lim_{{x \to 1}} \sin(f(x))$ is

(A) 0.909
(B) 0.841
(C) 0.141
(D) –0.416
(E) nonexistent
Free Response

23. (calculator allowed)

A continuous function $f$ is defined on the closed interval $-4 \leq x \leq 6$. The graph of $f$ consists of a line segment and a curve that is tangent to the $x$-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function $f$ is twice differentiable, with $f''(x) > 0$.

(a) Is $f$ differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.
24. (calculator not allowed)
   Let \( f \) be the function defined by
   \[
   f(x) = \begin{cases} \sqrt{x + 1} & \text{for } 0 \leq x \leq 3 \\ 5 - x & \text{for } 3 < x \leq 5. \end{cases}
   \]

   (a) Is \( f \) continuous at \( x = 3 \)? Explain why or why not.

   (c) Suppose the function \( g \) is defined by
   \[
   g(x) = \begin{cases} k\sqrt{x + 1} & \text{for } 0 \leq x \leq 3 \\ mx + 2 & \text{for } 3 < x \leq 5, \end{cases}
   \]
   where \( k \) and \( m \) are constants. If \( g \) is differentiable at \( x = 3 \), what are the values of \( k \) and \( m \)?
25. (calculator not allowed)

Let \( f \) be a function defined by \( f(x) = \begin{cases} 1 - 2\sin x, & x \leq 0 \\ e^{-4x}, & x > 0. \end{cases} \)

(a) Show that \( f \) is continuous at \( x = 0 \).

(b) For \( x \neq 0 \), express \( f'(x) \) as a piecewise-defined function. Find the value of \( x \) for which \( f''(x) = -3 \).
26. (calculator not allowed) 2012 AB 4

The function \( f \) is defined by \( f(x) = \sqrt{25 - x^2} \) for \(-5 \leq x \leq 5\).

(c) Let \( g \) be the function defined by \( g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x + 7 & \text{for } -3 < x < 5. \end{cases} \)

Is \( g \) continuous at \( x = -3 \)? Use the definition of continuity to explain your answer.
The function $f$ is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 5$.

(c) Let $g$ be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x + 7 & \text{for } -3 < x < 5. \end{cases}$

Is $g$ continuous at $x = -3$? Use the definition of continuity to explain your answer.

$$\lim_{x \to -3^-} g(x) = \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \sqrt{25 - x^2} = 4$$

$$\lim_{x \to -3^+} g(x) = \lim_{x \to -3^+} (x + 7) = 4$$

Therefore, $\lim_{x \to -3} g(x) = 4$.

$g(-3) = f(-3) = 4$

So, $\lim_{x \to -3} g(x) = g(-3)$.
(c) Let \( g \) be the function defined by \( g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x + 7 & \text{for } -3 < x \leq 5 \end{cases} \)

Is \( g \) continuous at \( x = -3 \)? Use the definition of continuity to explain your answer.

\[
\lim_{{x \to -3^-}} g(x) = \lim_{{x \to -3}} \left( \frac{1}{25-x^2} \right) = \frac{1}{25-(-3)^2} = \frac{1}{25-9} = \frac{1}{16} = 1
\]

\[
\lim_{{x \to -3^+}} g(x) = \lim_{{x \to -3}} (x+7) = (-3+7) = 4
\]

Yes, \( g \) is continuous at \( x = -3 \). The left- and right-hand limits of \( g \) are equal to the definition of \( g \) at \( x = -3 \).

(d) Find the value of \( \int_0^5 \sqrt{25-x^2} \, dx \).

\[
\int_0^5 \sqrt{25-x^2} \, dx = -\frac{1}{2} \int_0^5 U^\frac{1}{2} \, dU = \frac{1}{2} \int_0^5 U^\frac{1}{2} 
\quad \text{with } U = 25 - x^2
\quad \text{at } U = 0
\quad \text{and } U = 25
\]

\[
\frac{1}{2} \left[ \frac{2}{3} U^{\frac{3}{2}} \right]_0^{25} = \frac{1}{2} \left[ \frac{2}{3} (25)^{\frac{3}{2}} - \frac{2}{3} (0)^{\frac{3}{2}} \right] = \frac{1}{2} \left[ \frac{2}{3} (125) - 0 \right] = \frac{250}{3}
\]

\[
U = \sqrt{25-x^2} \quad \text{and } \quad du = -2x \, dx
\]

\[
U(5) = 25 - (5)^2 = 0
\]

\[
U(5) = 25 - (0)^2 = 25
\]
(c) Let \( g \) be the function defined by \( g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x+7 & \text{for } -3 < x \leq 5. \end{cases} \)

Is \( g \) continuous at \( x = -3 \)? Use the definition of continuity to explain your answer.

The definition of continuity at \( P(x, y) \) is

The \( \lim_{{x \to a}} g(x) \) for \( a \) is continuous at \( a + x = -3 \) means

for the two functions

that meet \( a + x = -3 \);

Since the two limits are the same, \( g(x) \) is continuous.

\[
\lim_{{x \to -3}} \sqrt{25-x^2} = 4
\]

\[
\lim_{{x \to -3}} x + 7 = 4
\]

(d) Find the value of \( \int_{0}^{5} x\sqrt{25-x^2} \, dx \).

\[
\int_{0}^{5} x\sqrt{25-x^2} \, dx = \frac{25-x^2}{2} \bigg|_{0}^{5} = \frac{1}{2} (25 - 5^3) = \frac{1}{2} (-125) = -\frac{125}{2}
\]

\[
\int_{0}^{5} x\sqrt{25-x^2} \, dx = \frac{125}{3}
\]

\[
\int_{0}^{5} x\sqrt{25-x^2} \, dx = 125/3
\]
(c) Let \( g \) be the function defined by \( g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x + 7 & \text{for } -3 < x \leq 5. \end{cases} \)

Is \( g \) continuous at \( x = -3 \)? Use the definition of continuity to explain your answer.

\[
\lim_{x \to -3^+} x + 7 = 4
\]
\[
\lim_{x \to -3^-} \sqrt{25 - x^2} = 4
\]

Since the \( \lim_{x \to -3^+} g(x) = \lim_{x \to -3^-} g(x) \), the \( \lim_{x \to -3} g(x) \) exists. Since \( \lim_{x \to -3} g(x) = g(-3) \) and \( g(-3) \) exists, \( g \) is continuous at \( x = -3 \).

(d) Find the value of \( \int_{0}^{5} x\sqrt{25 - x^2} \, dx \).

\[
\int_{0}^{5} x\sqrt{25 - x^2} \, dx = \int_{0}^{5} (25 - u)^{1/2} \, du
\]

\[
= \int_{0}^{5} \left( \frac{2}{3} u^{3/2} \right) \, du
\]

\[
= \left[ \frac{2}{3} \left( \frac{2}{3} u^{3/2} \right) \right]_{0}^{5}
\]

\[
= \left[ \frac{2}{3} \left( \frac{2}{3} (5)^{3/2} \right) \right] - \left[ \frac{2}{3} \left( \frac{2}{3} (0)^{3/2} \right) \right]
\]

\[
= \frac{2}{3} \left( \frac{2}{3} (5\sqrt{5}) \right)
\]

\[
= \frac{4}{9} \cdot 5\sqrt{5}
\]

\[
= \frac{20\sqrt{5}}{9}
\]
(c) Let \( g \) be the function defined by \( g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x + 7 & \text{for } -3 < x \leq 5. \end{cases} \)

Is \( g \) continuous at \( x = -3 \)? Use the definition of continuity to explain your answer.

\[
\sqrt{25 - x^2} = x + 7 \\
\sqrt{25 - (-2)^2} = (-3) + 7
\]

\[4 = 4\]

\( g \) is continuous at \( x = -3 \) because the functions that meet at \( x = -3 \) have the same value at \( x = -3 \).

\[
\ell(-3) = (-3) + 7
\]

(d) Find the value of \( \int_0^4 x\sqrt{25 - x^2} \, dx \).

\[
\left(-\frac{1}{2}\right) \int_0^5 x\sqrt{25 - x^2} \, dx = \frac{375}{2}
\]

\[
\frac{25}{2} \left[ x \sqrt{25 - x^2} \right]_0^5 - \frac{25}{2} \left[ \frac{1}{2} (25 - x^2)^{3/2} \right]_0^5
\]

\[
-\frac{1}{2} \left( 0 - 375 \right)
\]

\[
= 187.5
\]
(c) Let \( g \) be the function defined by \( g(x) = \begin{cases} f(x) & \text{for } -5 \leq x \leq -3 \\ x + 7 & \text{for } -3 < x \leq 5. \end{cases} \)

Is \( g \) continuous at \( x = -3 \)? Use the definition of continuity to explain your answer.

\[
\sqrt{25 - (-1)^2} - (8) + 7
\]

\[ y = 4 \]

\( g \) is continuous at \( x = -3 \) because the \( y \) value at \( x = -3 \) for both functions is 4.

This shows that the function doesn't have to break when transitioning from \( f(x) \) in \( x + 7 \), and that \[
\lim_{{x \to -3^+}} g(x) = \lim_{{x \to -3^-}} g(x).
\]

(d) Find the value of \( \int_0^5 x \sqrt{25 - x^2} \, dx \).

Let \( 25 - x^2 = u \)

\[
du = -2x \, dx
\]

\[
\frac{1}{2} \, du = x \, dx
\]

\[
\frac{1}{2} \int_0^5 \sqrt{u} \, du
\]

\[
\frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right]_0^{25}
\]

\[
= \frac{1}{3} \left[ \frac{2}{3} (25)^{3/2} - \frac{2}{3} (0)^{3/2} \right]
\]

\[
= \frac{1}{3} \left[ \frac{2}{3} (125) - \frac{2}{3} (0) \right]
\]

\[
\frac{1}{3} \left[ \frac{2}{3} (125) \right]
\]

\[
= \frac{1}{3} \left[ \frac{250}{3} \right]
\]

\[
= \frac{250}{9}
\]

\[
156.25
\]

\[
156.25
\]