Limits, Continuity, and Differentiability Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

Quick Check for Understanding: Student graphs will vary. Sample answers given.

1. Sketch a function with the property that $f(a)$ exists but $\lim_{x \to a} f(x)$ does not exist.

2. Sketch a function with the property that $\lim_{x \to a} f(x)$ exists but $f(a)$ does not exist.

3. Sketch a function with the property that $f(a)$ exists and $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$.
Multiple Choice

1. D (1985 AB5)
   Since the limit is taken as \( n \to \infty \) and the exponents in the numerator and denominator are equal, use the ratio of the leading coefficients to find that the limit is \( \frac{4n^2}{n^2} = 4 \).

2. B (2008 AB1)
   Multiplying in the numerator and denominator yields the equivalent limit:
   \[
   \lim_{x \to \infty} \frac{-2x^2 + 7x - 3}{x^3 + 2x - 3} = -2.
   \]

   This limit represents the definition of the derivative of the function, \( f(x) = 8x^8 \) at \( x = \frac{1}{2} \).
   \[
   f'(x) = 64x^7; \quad f'(\frac{1}{2}) = \frac{1}{2}.
   \]

4. D (1985 BC29 appropriate for AB)
   Let \( x - \frac{\pi}{4} = t \).
   \[
   \lim_{x \to \frac{\pi}{4}} \frac{\sin \left( x - \frac{\pi}{4} \right)}{x - \frac{\pi}{4}} = \lim_{t \to 0} \frac{\sin t}{t} = 1.
   \]

5. B (1988 AB29)
   This limit represents the derivative of the function, \( f(x) = \tan 3x \). Using the chain rule,
   \[
   f'(x) = 3 \sec^2(3x).
   \]

6. C (1993 BC2 appropriate for AB)
   \[
   \lim_{x \to 0} \frac{2x^2 + 1 - 1}{x^2} = 2
   \]

   Since \( f'(x) = \cos x \), \( f(x) = \sin x \). Also \( f(0) = 0 \), \( g(0) = 0 \), and \( g'(x) = 1 \), hence
   \[
   \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\sin x}{x} = 1.
   \]
   Alternatively, by L’Hopital’s rule,
   \[
   \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{f'(0)}{g'(0)} = \frac{\cos 0}{1} = 1.
   \]

8. C (1993 AB29)
   \[
   \lim_{x \to 0} \frac{1 - \cos \theta}{2 \sin^2 \theta} = \lim_{x \to 0} \frac{1 - \cos \theta}{2(1 - \cos^2 \theta)} = \lim_{x \to 0} \frac{1 - \cos \theta}{2(1 - \cos \theta)(1 + \cos \theta)} = \lim_{x \to 0} \frac{1}{2(1 + \cos \theta)} = \frac{1}{4}.
   \]
   Alternatively, by L’Hopital’s rule,
   \[
   \lim_{x \to 0} \frac{1 - \cos \theta}{2 \sin^2 \theta} = \lim_{x \to 0} \frac{\sin \theta}{4 \sin \theta \cos \theta} = \lim_{x \to 0} \frac{1}{4 \cos \theta} = \frac{1}{4}.
   \]
9. E (1985 AB41)
Using the given limit, there is not enough information to establish that \( f'(a) \) exists, nor that \( f(x) \) is continuous or defined at \( x = a \), nor that \( f(a) = L \). For example, consider the function whose graph is the horizontal line \( y = 2 \) with a hole at \( x = a \). For this function \( \lim_{x \to a} f(x) = 2 \) and none of the given statements are true.

10. E (2003 AB3)
By definition of a horizontal asymptote, E is correct.

11. E (1993 AB35)
Since \( y = 2 \) is a horizontal asymptote, the ratio of the leading coefficients must be 2; therefore, \( a = 2 \). Since there is a vertical asymptote at \( x = -3 \), set the denominator, \(-3 + c\), equal to 0, so \( c = 3 \) then \( a + c = 2 + 3 = 5 \).

\[ f(3) = 6(3) - 9 = 9 \]
\[ \lim_{x \to 3} x^2 = \lim_{x \to 3} 6x - 9 = 9 \]
Since \( f(3) = \lim_{x \to 3} f(x) \) the function is continuous at \( x = 3 \)
\[ f'(x) = \begin{cases} 
2x, & x < 3 \\
6, & x > 3 
\end{cases} \]
and \( \lim_{x \to 3^-} 2x = \lim_{x \to 3^+} 6 = 6 \)
Since the left and right limits of the derivative of the function are equivalent from either side of 3, the function is differentiable at \( x = 3 \).

13. D (2003 AB20)
Using substitution, the one sided limits as \( x \to 3 \) are both equal to 5; therefore, I and II are true. \( f \) is not differentiable at \( x = 3 \) since \( f'(3) = 1 \) for \( x \leq 3 \) and \( f'(3) = 4 \) for \( x > 3 \).

Use the top piece of the piecewise function for the limit since the bottom piece gives the value of \( f(2) \). Using factoring, \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2} x + 2 = 4 \). Since this limit does not equal \( f(2) = 1 \), the function \( f \) is not continuous or differentiable at \( x = 2 \).

15. C (1988 BC5 appropriate for AB)
\[ \lim_{x \to a} f(x) = f(a) \] for all values of \( a \) except 2. \[ \lim_{x \to 2} f(x) = \lim_{x \to 2} (x-3) = 0 \neq -1 = f(2) \].
Thinking graphically, the absolute value function will have a sharp corner at 3, thus the derivative does not exist at that point. The question can also be worked algebraically as follows:
The one sided limits are not equal:
\[
\lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3} = \frac{2 + |x - 3| - 2}{x - 3} = -1
\]
\[
\lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} = \frac{2 + |x - 3| - 2}{x - 3} = 1.
\]
Therefore, the value of \( f'(3) \) does not exist.

17. A (1993 AB5)
Consider \( f(a) = \lim_{x \to a} f(x) \)
Use factoring to simplify the function and then substitute for \( x \):
\[
\lim_{x \to 2} \frac{x^2 - 4}{x + 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x + 2} = x - 2; \text{ therefore, } f(-2) = -4.
\]

18. B (1997 AB15)
The left and right limits as \( x \to a \) are both equal to 2. The limit as \( x \to b \) does not exist since the one-sided limits are not equivalent; therefore, A, C and D cannot be true either.

19. A (2003 AB13/BC13)
The graph of \( f \) is continuous at \( x = a \); however, since the graph has a sharp turn at \( x = a \), the function is not differentiable at \( x = a \).

The one-sided limits as \( x \to 4 \) are equivalent for the graphs of \( f \) in I and II but not for III.

21. C (2008 AB77)
\[
\lim_{x \to 2^-} f(x) \text{ and } \lim_{x \to 2^+} f(x) \text{ exist; however, since they are not equivalent, the } \lim_{x \to 2} f(x) \text{ does not exist.}
\]

22. A (2003 BC81 appropriate for AB)
\[
\sin(2) \approx 0.9093
\]
Free Response

23. 2009B AB3abc/BC3abc

(a) \[
\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \frac{2}{3}, \\
\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} < 0
\]
Since the one-sided limits do not agree, \( f \) is not differentiable at \( x = 0 \).

24. 2003 AB6ac

(a) \( f \) is continuous at \( x = 3 \) because
\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2.
\]
Therefore, \( \lim f(x) = 2 = f(3) \).

(c) Since \( g \) is continuous at \( x = 3 \),
\[
2k = 3m + 2.
\]
\[
g'(x) = \begin{cases} 
\frac{k}{2\sqrt{x+1}} & ; 0 < x < 3 \\
\frac{m}{4} & ; 3 < x < 5
\end{cases}
\]
\[
\lim_{x \to 3^-} g'(x) = \frac{k}{4} \quad \text{and} \quad \lim_{x \to 3^+} g'(x) = m
\]
Since these two limits exist and \( g \) is differentiable at \( x = 3 \), the two limits are equal. Thus \( \frac{k}{4} = m \).
\[
8m = 3m + 2; \quad m = \frac{2}{5} \quad \text{and} \quad k = \frac{8}{5}
\]
25. 2011 AB6ab

(a) \( \lim_{x \to 0} (1 - 2\sin x) = 1 \)
\( \lim_{x \to 0} e^{-4x} = 1 \)
\( f(0) = 1 \)
So, \( \lim_{x \to 0} f(x) = f(0) \).
Therefore \( f \) is continuous at \( x = 0 \).

(b) \( f'(x) = \begin{cases} -2\cos x & \text{for } x < 0 \\ -4e^{-4x} & \text{for } x > 0 \end{cases} \)
\(-2\cos x \neq -3 \) for all values of \( x < 0 \).
\(-4e^{-4x} = -3 \) when \( x = -\frac{1}{4}\ln\left(\frac{3}{4}\right) > 0 \).
Therefore \( f'(x) = -3 \) for \( x = -\frac{1}{4}\ln\left(\frac{3}{4}\right) \).

26. 2012 AB 4c

(c) \( \lim_{x \to -3} g(x) = \lim_{x \to -3} f(x) = \lim_{x \to -3} \sqrt{25 - x^2} = 4 \)
\( \lim_{x \to -3^+} g(x) = \lim_{x \to -3^+} (x + 7) = 4 \)
Therefore, \( \lim_{x \to -3} g(x) = 4 \).
\( g(-3) = f(-3) = 4 \)
So, \( \lim_{x \to -3} g(x) = g(-3) \).