Chapter 1 Answers

Lesson 1.1

1. Different ranking methods will probably not rank the same soft drink first unless the soft drink received a majority of first-place votes. That is, if more than half the members of the class ranked the same soft drink first then any reasonable ranking method will rank it first. (Note that experience has shown that many students adopt some version of a point system in this activity. The most common ways that students assign points are 5 for first, 4 for second, and so forth—giving the win to the soft drink with the highest total or by reversing this scheme and giving the win to the soft drink with the lowest total. These two schemes are equivalent. If the "spacing" between the points is unequal—6 for first, 4 for second, 3 for third, 2 for fourth, and 1 for fifth, for example—the group ranking may differ.)

2–5. Similar statements hold for second through fifth. For example, different ranking methods will probably not all agree on the second-place soft drink unless that soft drink has a strong second-place showing in the class.

6. A sample answer for a method that assigns 5 points to first, 4 to second, and so forth can be found in the answer to Exercise 13b of Lesson 1.2.

7. For example, two groups may have used a point system, but assigned the points differently. See the answer to Exercise 1 above for a discussion of whether two point systems should produce the same results.

8. A tie can be broken simply by flipping a coin. However, it can also be broken by modifying the method. If, for example, a point system results in a tie, the win can be given to the soft drink with the most firsts. Alternately, the soft drinks that are tied could be compared and the win given to the one that is ranked above the other more often.

9. a. For example, if the method assigns 4 points to first, 3 to second, and so forth, the results are A: 50, B: 83, C: 69, D: 58. B is the winner.

10. For example, ranking candidates would require the use of a group-ranking method (a point method, for example) that considers voter’s preferences for all candidates. However, ranking could take voters more time, resulting in longer waits at the polls—planning one’s vote before going to the polls would be essential.

11. a. 120 720

   b. The number of schedules possible when there are \( n \) choices is \( n \) times the number of schedules possible when there are \( n - 1 \) choices.

12. 8 12 17

13. 25 53 109

14. Sample answer: To get a number in a list, multiply the previous number by 3, and then subtract 4.
Lesson 1.2

1. Students should choose the soft drink that was ranked first by the most students. They should call it a majority winner only if over half the class ranked it first.

2. Students should give each soft drink 5 points for each first-place ranking it received, 4 points for each second-place ranking, and so forth. The winner is the soft drink with the highest point total.

3. Students should eliminate the three soft drinks with the fewest first-place votes. All first-place votes for these soft drinks should be transferred to the remaining soft drink with the higher ranking. The winner is the soft drink with the higher vote total.

4. Students should eliminate the soft drink with the fewest first place votes. For each voter who ranked this soft drink first, the vote should be transferred to the voter’s second choice. The votes are re-totaled and the process repeated until only two remain. The winner is the soft drink with the higher vote total.

5. a. Sample answer: Salzburg got the fewest votes on the first round and was eliminated. On the second round, Sochi got most of Salzburg’s votes and defeated Pyeongchang. Apparently three committee members who voted on the second round did not vote on the first round

   b. |          | Rio de Janeiro | Madrid | Tokyo |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Vote</td>
<td>26</td>
<td>28</td>
<td>22</td>
</tr>
<tr>
<td>Percentage</td>
<td>30.8% (8/26)</td>
<td>19.2% (5/26)</td>
<td>23.1% (6/26)</td>
</tr>
</tbody>
</table>

6. a. A 30.8% (8/26) 69.2% (18/26)
   B 19.2% (5/26)  0.0% (0/26)
   C 23.1% (6/26)  0.0% (0/26)
   D 26.9% (7/26)  30.8% (8/26)

   b. For example, A can be considered the most objectionable because a majority of voters ranked A last. By similar reasoning, B and C could be considered least objectionable.

   c. For example, B could be considered the best choice because all voters ranked B either first or second.

   d. For example, B has the fewest first-place rankings.

7. a. 43.6%

   b. The winner must receive slightly more than a third of the vote: at least 34%, to the nearest whole number; at least 33.4% to the nearest tenth.
8. Plurality: B because B has the most first-place rankings.
   Borda: D because D's point total (153) is highest (A has 130, B has 139, C has 128).
   Runoff: C (B and C are in the runoff; C wins 35–20).
   Sequential runoff: C (D and A are eliminated first; A's 16 votes are transferred to C; C defeats B 35–20).

9. a. Second place can be given to the choice with the second-highest vote total, and so forth.
   A is first with 8 first-place votes, D is second with 7, C is third with 6, and B is fourth with 5.

b. After the runoff model has determined a winner, it can be repeated with the winning choice removed.
   Remove D from all schedules. A now has 8 firsts, B has 12 and C has 6. C is eliminated and B defeats A for second by 18 to 8. B is removed from all schedules and C defeats A by 18 to 8. Thus, D is first, B is second, C is third, and A is fourth.

c. For example, the choice that is eliminated first can be ranked last, the choice that is eliminated second can be ranked next-to-last, and so forth.
   With this model, the ranking for the example is C, A, D, B.

10. a. 3 points for first, 2 for second, and 1 for third.

b. Yes. The ranking would be Manziel, Te'o, Klein, Lee, Austin, Clowney, Miller and Lynch (tie), Barner and Jones (tie).

11. For example, in the U.S. House race, the number of voters who cast votes in the runoff is much smaller than the number of voters who voted in the regular election. Since an instant runoff does not require voters to return to the polls, the problem of low turnout in runoffs is avoided.

12. \( C_n = C_{n-1} - 1 \)
13. a. Sample answer:
   1. Total all the first-place rankings.
   2. Eliminate the choice with the fewest first-place rankings. Transfer those votes to the remaining choice that is ranked next highest.
   3. Repeat steps 1 and 2 until a single choice remains.

   b. Sample answer:
   1. For each choice, total the number of first-, second-, third-place rankings, and so forth.
   2. For each choice, multiply the number of first-place rankings by the number of choices, the number of second-place rankings by one less than the number of choices, the number of third-place rankings by two less than the number of choices, and so forth.
   3. Rank the choices according to the order of the point totals calculated in step 2.

14. a. [142 111 158 159]

   b. You can determine the plurality winner and the Borda winner. You cannot determine the runoff and sequential runoff winner because you cannot determine the voters' rankings and, therefore, you cannot transfer votes.
Lesson 1.3

1. In some cases, the class data will not produce a Condorcet winner. Students should check to see that the winner they propose can beat each of the other soft drinks in a one-on-one election. (Note that if the class ballot produced a majority winner, the majority winner is the Condorcet winner.)

2. One possibility is to give the election to the choice that wins the most one-on-one contests. Another is to resort to some other method.

3. a. A and B because A would defeat B, but A would lose to C.
   b. Yes. If B and C are first, B wins, but loses to A in the second round.

4. a. A with 242 points (B has 238, C has 120)
   b. B with 238 points (A has 204, C has 158)

5. Because of political polls and media coverage

6. Sample answer:

   ![Diagram of rankings]

   The plurality winner is A. The supporters of B would be tempted to vote for their second-choice, C, in order to prevent A from winning.

7. a. A, the plurality winner, is ranked last by 69.2% (45/65) of the voters.
   b. The runoff winner is C. C is an improvement over A. However, a majority of the voters (35/65 or 53.8%) rank C next-to-last.
   c. The sequential runoff winner is also C, so in this case it cannot be considered an improvement over the runoff winner.

8. a. A defeats B by 68 to 32.
   b. C defeats A by 55 to 45.
   c. Because you would not expect the winning choice to lose if its standing improves
9. a. A (280), B (260), C (190), D (170)

b. With C removed:

\[\text{With a 3–2–1 Borda count: B (220), A (190), D (130)}\]

c. Because the group ranking of teams A, B, and D, changes when team C is removed.

10. Sample answer:

   The plurality model gives the win to the choice with the most first-place votes. This model can be unfair because the winner can be ranked last by a majority of the voters, as shown in Exercise 7.

   The Borda model awards points for all rankings, not just first. However, as shown in Exercise 9, the Borda model can produce a different winner when a non-winning choice is removed. The Borda model can also encourage insincere voting, as shown in Exercise 4. (Any of the models under consideration here can encourage insincere voting.)

   The runoff model eliminates all but two choices and holds a new election by transferring votes from the eliminated choices to those that remain. Sequential runoff is similar, but it eliminates only one choice at a time, transferring votes accordingly. If voters rank the candidates, these runoffs can be conducted without voters returning to the polls. As shown in Exercise 8, runoff models can produce an unusual paradox in which a winning choice actually loses by getting more votes.

   The Condorcet model requires that the winner be capable of beating each of the other choices in a one-on-one election. However, the Condorcet model sometimes does not produce a winner. It can also produce a winner that no voters ranked first.
11. a. There are two new comparisons. A total of three comparisons must be made.
   b. There are three new comparisons. A total of six comparisons must be made.
   c.  
   

   
12. \( C_n = C_{n-1} + n - 1 \)

13. a. Sample answers: A wins against B, B wins against C, and A wins against C. A wins against E, E wins against F, but A does not win against F.
   b. A is first with 4 points, F is second with 3 points, and the other four teams are tied with 2 points each.
   c. Sample answer: Since this system seems to result in frequent ties, modify it by holding a runoff between the top two. In this case, A and F are in the runoff. F wins the runoff because it wins the comparison with A.
   d. With D out, A and F are tied for first and second with 3 points each. E is next with 2 points, followed by B and C with 1 point each.
Lesson 1.4

1. Nondictatorship (the teacher’s pet is the dictator, not the teacher)

2. If the method was repeated, the same ranking might not result. Therefore, condition 5 is violated. Nondictatorship (condition 1) is also violated.

3. None require a secret ballot. A secret ballot is not desirable in a democratic representative assembly such as the U.S. Congress.

4. Condition 4

5. Sample answer:

```
  A  C  B
B   B  C
C  A  A
  20 15  6
```

The plurality winner is A. If B (the irrelevant alternative) is removed, C is the plurality winner.

6. Condition 4 is closest because the existence of C on the ballot may change the way a person decides to rank A and B.

7. None. One of the two must be a majority winner (excluding the remote possibility of a tie).

8. a. A is a plurality winner and a Borda winner. B is a runoff winner and a sequential runoff winner. There is no Condorcet winner.

b. If C withdraws, A becomes a runoff winner and a sequential runoff winner instead of B. If D withdraws, B becomes a Borda winner instead of A. (Note: if D withdraws, B also becomes a Condorcet winner.)

9. a. No. If page A links to page B and page B links to page C, then page A does not necessarily have to link to page C.

b. Yes. If site A has more links to it than site B, and site B has more links to it than site C, then site A must have more links to it than site C. This is just the transitive property as it applies to the integers.

10. a–b. Students should select as the winner the soft drink that has the most votes. c–d. The answers depend on the class data.

11. No. Since the voters do not rank, they do not have to rate their first choice lower than another choice. They can choose to approve of more than one.

12. In both cases the effect is the same as not voting.
13. a. Assuming the second and third choices get votes about half the time, the results might be:
   A: 22
   B: 11 + 10 + 9 = 30
   C: 11 + 9 = 20
   D: 20
   E: 18 + 10 = 28

   b. Since the strongest choices, B and E, get much of their support from random votes and since it is a close race, it is possible that a few flips of the coin could determine the election. Thus, the election becomes random, which seems to violate the spirit of Arrow’s fifth condition.

14. a. \{ \} \{A\} \{B\} \{C\} \{A, B\} \{A, C\} \{B, C\} \{A, B, C\}


   c. \(V_n = 2V_{n-1}\)

15. Sample answer:
   1. Apply the two-choice algorithm.
   2. Write the list in step 1 of the two-choice algorithm in a new list 1.
   3. Select the first pair produced by the two-choice algorithm and draw a line through the choices in list 1 that are in this pair or that come before either member of the pair. Write this pair in list 2.
   4. Select the first choice in list 1 that does not have a line through it and write it after the pair in list 2 to form a triple.
   5. Repeat steps 3 and 4 until each choice in list 1 has a line through it.
   6. Start a new list 1 by repeating step 2.
   7. Repeat step 3 by selecting the next pair produced by the two-step algorithm, then repeat steps 4 and 5.
   8. Repeat steps 6 and 7 until all pairs produced by the two-step algorithm have been used.

16. 4 5

17. \(V_{1n} = V_{1n-1} + 1\) or \(V_{1n} = \frac{n}{n-1} V_{1n-1}\)

18. 6 10

19. \(V_{2n} = V_{2n-1} + n - 1\) or \(V_{2n} = \frac{n}{n-2} V_{2n-1}\)
Lesson 1.5

1. a. The possible coalitions: { ; 0} {A; 3} {B; 2} {C; 1} {A, B; 5} {B, C; 3} {A, C; 4} {A, B, C; 6}
   The winning coalitions: {A, B; 5} {A, C; 4} {A, B, C; 6}
   b. A:3   B:1   C:1
   c. A:2   B:2   C:0

2. a. A has all the power since 7 votes are needed to pass an issue.
   b. A is a dictator. B and C are dummies.

3. If the members have 2, 2, and 1 votes, and 4 votes are needed to pass an issue, the first two members each have half the power, and the third has none. There are no other possibilities.

4. It is impossible since, for example, no distribution of votes will give the sophomores 45% of the power.

5. Yes, because it makes one of the coalitions less likely than the others. This increases the power of the sophomore representatives.

6. \( C_n = 2C_{n-1} \)

7. \{ \} \{A\} \{B\} \{C\} \{D\} \{A, B\} \{A, C\} \{A, D\} \{B, C\} \{B, D\} \{C, D\} \{A, B, C\} \{A, B, D\}
   \{A, C, D\} \{B, C, D\} \{A, B, C, D\}

8. a. The winning coalitions are: {A, B; 51%} {A, C; 51%} {A, B, C; 76%}
   \{A, B, D; 75%\} \{A, C, D; 75%\} \{B, C, D; 74%\} \{A, B, C, D; 100%\}. Of these, A is essential to 5, B to 3, C to 3, and D to 1.
   b. The winning coalitions are: {A, B; 88%} {A, C; 54%} {A, D; 52%} {A, B, C; 95%}
   \{A, B, D; 93%\} \{A, C, D; 59%\} \{B, C, D; 53%\} \{A, B, C, D; 100%\}. Of these, A is essential to 6, B to 2, C to 2, and D to 2.
   c. In part a, D has 24% of the stock and one-twelfth of the power.
   In part b, D has 5% of the stock, but two-twelfths of the power.

9. a. The three members with 31, 31 and 28 votes have equal power; the other three members have no power.
   b. Hempstead had two-thirds of the power and 57% of the population; Oyster Bay had one-third of the power and 22% of the population; the others had no power.

10. a. Sample answer: Voters A, B, C, and D have 4, 2, 1, and 1 votes, respectively, and 7 votes are needed to pass an issue. In the coalition \{A, B, C, D\}, voters A and B are essential, but C and D are not. There are two minimal winning coalitions: \{A, B, C\} and \{A, B, D\}.
   b. No. As the previous example shows, A, for example, is a member of two minimal winning coalitions and receives a power point from each of these. However A is also an essential member of the winning coalition \{A, B, C, D\}, for which A receives a third power point.
11. a. Sample answer: California has one elector per \( \frac{33871648}{54} \approx 627,253 \) people.
   
   Wyoming has one elector per \( \frac{493782}{3} \approx 164,594 \) people.
   
   b. Yes. If each state has two fewer electors, Gore has 225 votes and Bush has 211.
   
   c. Sample answer: Note that in the 2000 election, Bush won 30 states and Gore won 20 plus the District of Columbia. Perhaps the founders felt that in a very close election, the candidate who carried more states should win.

A note on Computer Exploration 12: With three voters there are only two systems that give each member at least some power (see Exercise 3). With four voters, there are eight different systems that give each voter at least some power.

A note on Project 13: The voting system can be described by giving 7 votes to each permanent member and 1 vote to each elected member, with 39 votes necessary to pass an issue. There are a total of 848 winning coalitions. The permanent members are essential to all of these. Each elected member is essential to 84. A permanent member has about 10 times the power of an elected member.
Chapter 1 Review

1. A reasonable summary should include the following points.
   - A summary of the plurality, Borda, runoff, sequential runoff, Condorcet and approval models, including at least one undesirable result that can occur with each
   - Arrow’s theorem
   - A discussion of weighted voting, including the fact the distributing power according to population is not equitable and the use of a power index to measure power

2. a. D (A: 186, B: 151, C: 221, D: 229, E: 158)
   b. B
   c. A (A defeats B in the runoff by 41 to 22)
   d. E (D is eliminated first and 9 votes are transferred to E; A is eliminated next and 20 votes are transferred to C; C is eliminated next and 20 votes are transferred to E).
   e. C
   f. C (totals are A: 20, B: 22, C: 32, D: 31, E: 21)

3. 19, 42, 89

4. a. It can occur in either a runoff or sequential runoff model.
   b. For example, the winner of an election can lose to another choice if a third choice withdraws.

5. a. Wilson; no
   b. They ranked him last.
   c. The voters in the last group could have switched to Roosevelt, their second choice, and thereby prevented Wilson from winning.
   d. Borda, runoff, and Condorcet give the election to Roosevelt.

6. Conditions 2, 3, and 5

7. Arrow’s fourth condition because Kwan is an irrelevant alternative in the judges’ decision between Bobek and Bonaly

8. Arrow proved that no group ranking model that ranks three or more choices will always adhere to his five fairness conditions.
9. Yes. Consider the following example:

If a 3-2-1 system is used, the results are A: 81, B: 88, C: 77. If a 4-2-1 system is used, the results are A: 101, B: 94, C: 92. In the first case, B wins; in the second, A wins.

10. a. Clinton: \(0.43 + 0.2 \times 0.38 + 0.35 \times 0.19 = 0.5725\) or about 57%
   
   Bush: \(0.38 + 0.15 \times 0.43 + 0.3 \times 0.19 = 0.5015\), or about 50%
   
   Perot: \(0.19 + 0.3 \times 0.43 + 0.2 \times 0.38 = 0.395\), or about 40%
   
   b. The total is approximately 147. Since each voter can cast more than one approval vote, the percentage of approval votes received can be more than 100. In other words, some voters vote for more than one candidate.

11. Sample answer: The Borda model gives the win to C. C seems a good choice because no voters rank C last, and 64% of the voters rank C either first or second.

12. This is a 5-3-1 Borda point system.

13. a. The plurality winner is A in each case.
   
   b. No. The runoff (and sequential runoff) winner changes from C to A when the preferences are reversed. The Borda winner changes from B to C. The Condorcet winner changes from B to A.
   
   c. In this set of preferences, runoff and sequential runoff methods demonstrate the reversal effect because C is a runoff and sequential runoff winner for the given preferences and the reversed. (When the preferences are reversed, plurality changes from A to C and Borda changes from D to C. There isn’t a Condorcet winner in either case.)

14. a. \([A, B, C, D; 12], [A, B, C; 10], [B, C, D; 8], [A, C, D; 9], [A, B, D; 9], [A, B; 7], [A, C; 7]\)
   
   b. A: 5, B: 3, C: 3, D: 1
   
   c. No, A’s power is disproportionately high, while D’s is low.
   
   d. All voters now have equal power.
15. The following table summarizes the winning coalitions, their vote totals, and their critical members.

<table>
<thead>
<tr>
<th>Coalition</th>
<th>Votes</th>
<th>Critical</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE</td>
<td>11</td>
<td>DE</td>
</tr>
<tr>
<td>CE</td>
<td>8</td>
<td>CE</td>
</tr>
<tr>
<td>CDE</td>
<td>13</td>
<td>E</td>
</tr>
<tr>
<td>BDE</td>
<td>12</td>
<td>DE</td>
</tr>
<tr>
<td>BCE</td>
<td>9</td>
<td>CE</td>
</tr>
<tr>
<td>BCD</td>
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<td>BCD</td>
</tr>
<tr>
<td>BCDE</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>ADE</td>
<td>12</td>
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<tr>
<td>ACE</td>
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<tr>
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<tr>
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<tr>
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<td>CD</td>
</tr>
<tr>
<td>ABCDE</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

The next table summarizes the power indices, the percentage of the power index total, and the percentage of the vote total.

<table>
<thead>
<tr>
<th>Member</th>
<th>Index</th>
<th>Index%</th>
<th>Vote%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>7.69</td>
<td>6.67</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>7.69</td>
<td>6.67</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>23.08</td>
<td>13.33</td>
</tr>
<tr>
<td>D</td>
<td>6</td>
<td>23.08</td>
<td>33.33</td>
</tr>
<tr>
<td>E</td>
<td>10</td>
<td>38.46</td>
<td>40</td>
</tr>
</tbody>
</table>

The indices of members A, B, and E approximately reflect their share of the total vote. Member C’s index is higher than it should be, at the expense of member D.