# Angles of Chords, Secants, and Tangents

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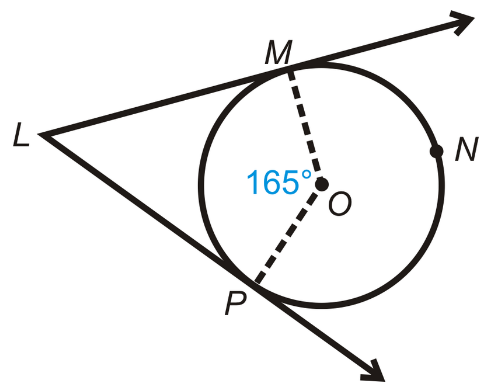
# CHAPTER 1 Angles of Chords, Secants, and Tangents

## Learning Objectives

- Find the measures of angles formed by chords, secants, and tangents.

## Review Queue




1. What is $m\angle OML$ and $m\angle OPL$? How do you know?
2. Find $m\angle MLP$.
3. Find $m\widehat{MNP}$.
4. Find $\frac{m\widehat{MNP}-m\widehat{MP}}{2}$. What is it the same as?

**Know What?** The sun's rays hit the Earth such that the tangent rays determine when daytime and night time are. The time and Earth's rotation determine when certain locations have sun. If the arc that is exposed to sunlight is $178^\circ$, what is the angle at which the sun's rays hit the earth $(x^\circ)$?

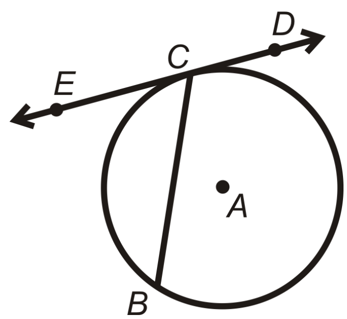

## Angle

When an angle is on a circle, the vertex is on the circumference of the circle. One type of angle *on* a circle is the inscribed angle, from the previous section. Recall that ***an inscribed angle is formed by two chords and is half the measure of the intercepted arc.*** Another type of angle *on* a circle is one formed by a tangent and a chord.

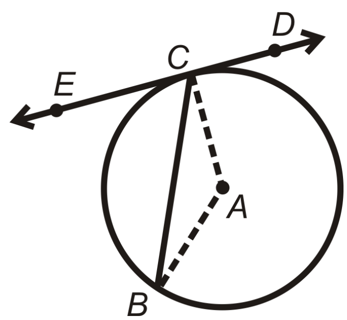
**Investigation 9-6: The Measure of an Angle formed by a Tangent and a Chord**

Tools Needed: pencil, paper, ruler, compass, protractor

1. Draw $\odot A$ with chord $\overline{BC}$ and tangent line $\overleftrightarrow{ED}$ with point of tangency $C$.




2. Draw in central angle $\angle CAB$. Then, using your protractor, find $m\angle CAB$ and $m\angle BCE$.




3. Find $m\widehat{BC}$ (the minor arc). How does the measure of this arc relate to $m\angle BCE$?

What other angle that you have learned about is this type of angle similar to?

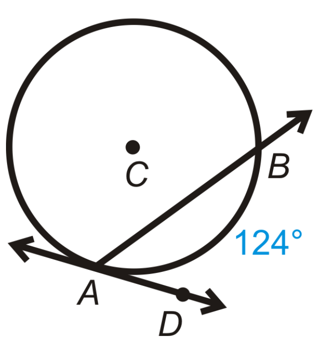
This investigation proves Theorem 9-11.

**Theorem 9-11:** The measure of an angle formed by a chord and a tangent that intersect on the circle is half the measure of the intercepted arc.

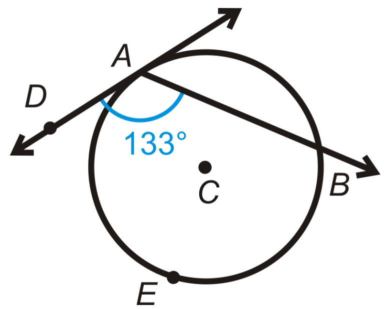
From Theorem 9-11, we now know that there are two types of angles that are half the measure of the intercepted arc; an inscribed angle and an angle formed by a chord and a tangent. Therefore, ***any angle with its vertex on a circle will be half the measure of the intercepted arc***.

**Example 1:** Find:

a) $m\angle BAD$




b) $m\widehat{AEB}$

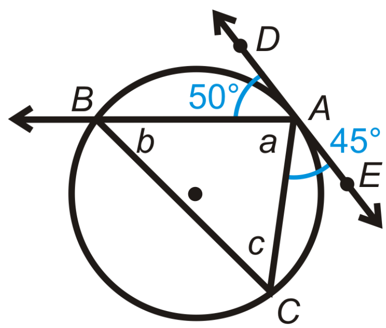



**Solution:** Use Theorem 9-11.

a) $m\angle BAD = \frac{1}{2} m\widehat{AB} = \frac{1}{2} \cdot 124^\circ = 62^\circ$

b) $m\widehat{AEB} = 2 \cdot m\angle DAB = 2 \cdot 133^\circ = 266^\circ$

**Example 2:** Find $a, b$, and $c$.




**Solution:** To find $a$, it is in line with $50^\circ$ and $45^\circ$. The three angles add up to $180^\circ$. $50^\circ + 45^\circ + m\angle a = 180^\circ, m\angle a = 85^\circ$.

$b$ is an inscribed angle, so its measure is half of $m\widehat{AC}$. From Theorem 9-11, $m\widehat{AC} = 2 \cdot m\angle EAC = 2 \cdot 45^\circ = 90^\circ$.

$m\angle b = \frac{1}{2} \cdot m\widehat{AC} = \frac{1}{2} \cdot 90^\circ = 45^\circ$.

To find $c$, you can either use the Triangle Sum Theorem or Theorem 9-11. We will use the Triangle Sum Theorem. $85^\circ + 45^\circ + m\angle c = 180^\circ, m\angle c = 50^\circ$.

From this example, we see that Theorem 9-8, from the previous section, is also true for angles formed by a tangent and chord with the vertex on the circle. If two angles, with their vertices *on* the circle, intercept the same arc then the angles are congruent.

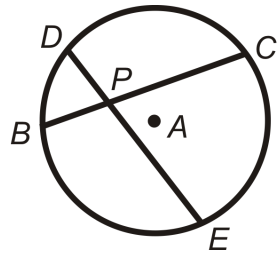
## Angles

An angle is considered *inside* a circle when the vertex is somewhere inside the circle, but not on the center. All angles inside a circle are formed by two intersecting chords.

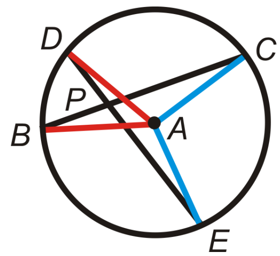
**Investigation 9-7: Find the Measure of an Angle *inside* a Circle**

Tools Needed: pencil, paper, compass, ruler, protractor, colored pencils (optional)

1. Draw $\odot A$ with chord $\overline{BC}$ and $\overline{DE}$. Label the point of intersection $P$.



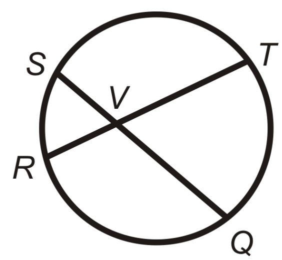

2. Draw central angles $\angle DAB$ and $\angle CAE$. Use colored pencils, if desired.




3. Using your protractor, find $m\angle DPB, m\angle DAB$, and $m\angle CAE$. What is $m\widehat{DB}$ and $m\widehat{CE}$?
4. Find $\frac{m\widehat{DB}+m\widehat{CE}}{2}$.
5. What do you notice?

**Theorem 9-12:** The measure of the angle formed by two chords that intersect *inside* a circle is the average of the measure of the intercepted arcs.

In the picture to the left:

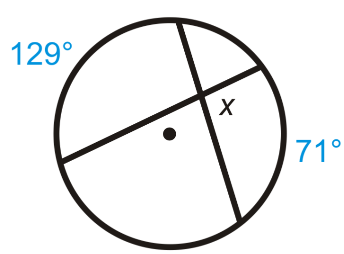



$$m\angle SVR = \frac{1}{2}\left(m\widehat{SR}+m\widehat{TQ}\right) = \frac{m\widehat{SR}+m\widehat{TQ}}{2} = m\angle TVQ$$
$$m\angle SVT = \frac{1}{2}\left(m\widehat{ST}+m\widehat{RQ}\right) = \frac{m\widehat{ST}+m\widehat{RQ}}{2} = m\angle RVQ$$

The proof of this theorem is in the review exercises.

**Example 3:** Find $x$.

a)



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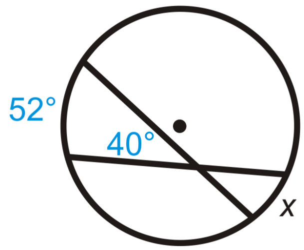


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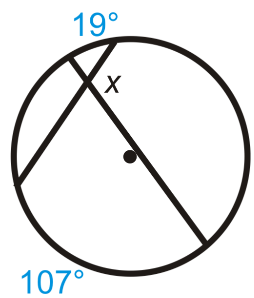



**Solution:** Use Theorem 9-12 and write an equation.

a) The intercepted arcs for $x$ are $129^\circ$ and $71^\circ$.

$$x = \frac{129^\circ + 71^\circ}{2} = \frac{200^\circ}{2} = 100^\circ$$

b) Here, $x$ is one of the intercepted arcs for $40^\circ$.

$$40^\circ = \frac{52^\circ + x}{2}$$
$$80^\circ = 52^\circ + x$$
$$38^\circ = x$$

c) $x$ is supplementary to the angle that the average of the given intercepted arcs. We will call this supplementary angle $y$.

$y = \frac{19^\circ + 107^\circ}{2} = \frac{126^\circ}{2} = 63^\circ$ This means that $x = 117^\circ; 180^\circ - 63^\circ$

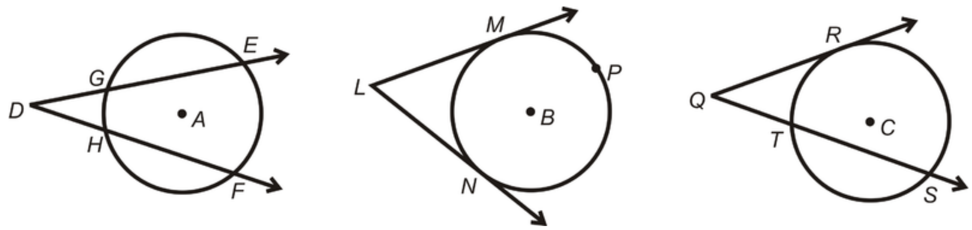
## Angles

An angle is considered to be outside a circle if the vertex of the angle is outside the circle and the sides are tangents or secants. There are three types of angles that are outside a circle: an angle formed by two tangents, an angle formed by a tangent and a secant, and an angle formed by two secants. Just like an angle inside or on a circle, an angle outside a circle has a specific formula, involving the intercepted arcs.

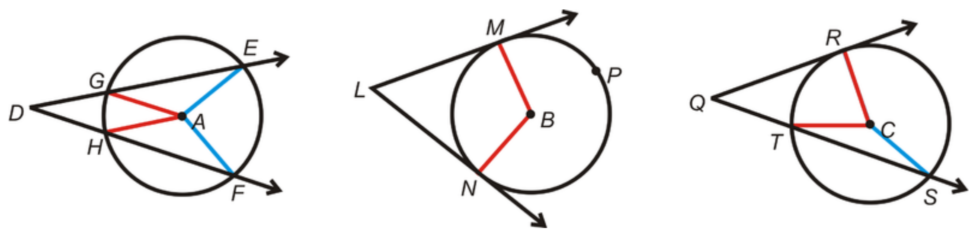
**Investigation 9-8: Find the Measure of an Angle outside a Circle**

Tools Needed: pencil, paper, ruler, compass, protractor, colored pencils (optional)

1. Draw three circles and label the centers $A, B$, and $C$. In $\odot A$ draw two secant rays with the same endpoint, $\overrightarrow{DE}$ and $\overrightarrow{DF}$. In $\odot B$, draw two tangent rays with the same endpoint, $\overrightarrow{LM}$ and $\overrightarrow{LN}$. In $\odot C$, draw a tangent ray and a secant ray with the same endpoint, $\overrightarrow{QR}$ and $\overrightarrow{QS}$. Label the points of intersection with the circles like they are in the pictures below.




2. Draw in all the central angles: $\angle GAH, \angle EAF, \angle MBN, \angle RCT, \angle RCS$. Then, find the measures of each of these angles using your protractor. Use color to differentiate.

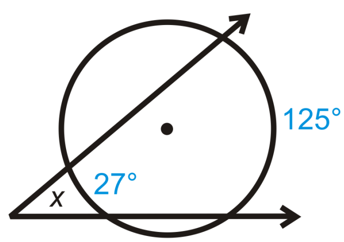



3. Find $m\angle EDF, m\angle MLN$, and $m\angle RQS$.
4. Find $\frac{m\widehat{EF}-m\widehat{GH}}{2}, \frac{m\widehat{MPN}-m\widehat{MN}}{2}$, and $\frac{m\widehat{RS}-m\widehat{RT}}{2}$. What do you notice?

**Theorem 9-13:** The measure of an angle formed by two secants, two tangents, or a secant and a tangent drawn from a point outside the circle is equal to half the difference of the measures of the intercepted arcs.

**Example 4:** Find the measure of $x$.

a)

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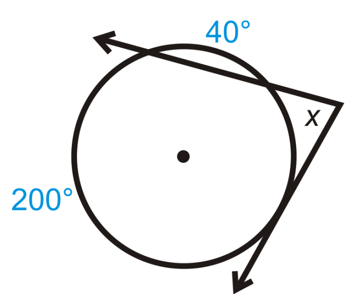


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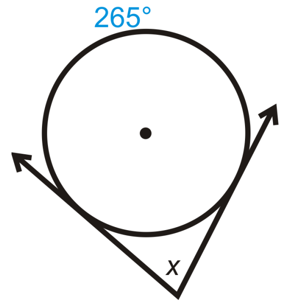


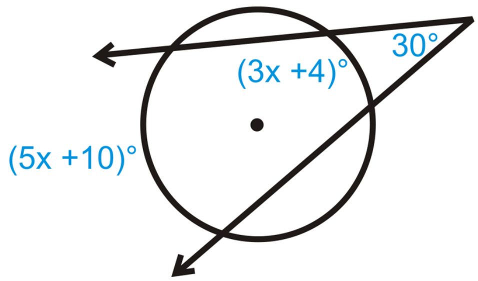

**Solution:** For all of the above problems we can use Theorem 9-13.

a) $x = \frac{125^\circ - 27^\circ}{2} = \frac{98^\circ}{2} = 49^\circ$

b) $40^\circ$ is not the intercepted arc. Be careful! The intercepted arc is $120^\circ$, $(360^\circ - 200^\circ - 40^\circ)$. Therefore, $x = \frac{200^\circ - 120^\circ}{2} = \frac{80^\circ}{2} = 40^\circ$.

c) First, we need to find the other intercepted arc, $360^\circ - 265^\circ = 95^\circ$. $x = \frac{265^\circ - 95^\circ}{2} = \frac{170^\circ}{2} = 85^\circ$

**Example 5:** ***Algebra Connection*** Find the value of $x$. You may assume lines that look tangent, are.




**Solution:** Set up an equation using Theorem 9-13.

$$\begin{aligned}
\frac{(5x+10)^\circ - (3x+4)^\circ}{2} &= 30^\circ \\
(5x+10)^\circ - (3x+4)^\circ &= 60^\circ \\
5x+10^\circ - 3x - 4^\circ &= 60^\circ \\
2x+6^\circ &= 60^\circ \\
2x &= 54^\circ \\
x &= 27^\circ
\end{aligned}$$

**Know What? Revisited** If $178^\circ$ of the Earth is exposed to the sun, then the angle at which the sun's rays hit the Earth is $2^\circ$. From Theorem 9-13, these two angles are supplementary. From this, we also know that the other $182^\circ$ of the Earth is not exposed to sunlight and it is probably night time.

## Review Questions

1. Draw two secants that intersect:
   a. inside a circle.
   b. on a circle.
   c. outside a circle.

2. Can two tangent lines intersect inside a circle? Why or why not?
3. Draw a tangent and a secant that intersect:
    a. on a circle.
    b. outside a circle.

Fill in the blanks.

4. If the vertex of an angle is on the _______________ of a circle, then its measure is _______________ to the intercepted arc.
5. If the vertex of an angle is _______________ a circle, then its measure is the average of the _____________-_____ arcs.
6. If the vertex of an angle is ________ a circle, then its measure is ______________ the intercepted arc.
7. If the vertex of an angle is ____________ a circle, then its measure is ___________ the difference of the intercepted arcs.

For questions 8-19, find the value of the missing variable(s).

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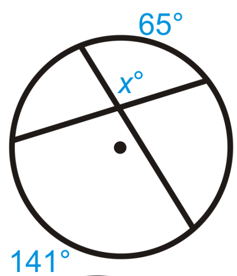


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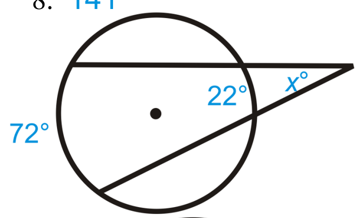


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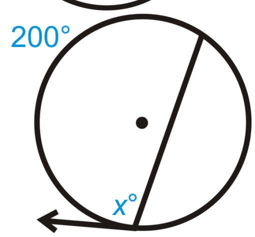


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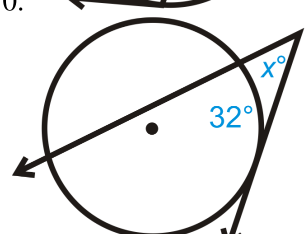


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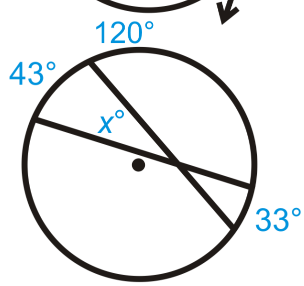
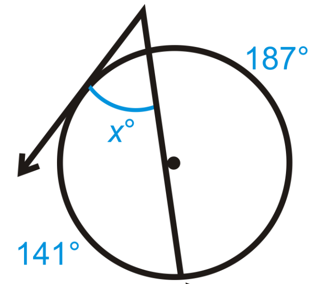
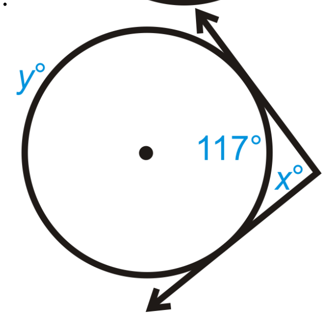
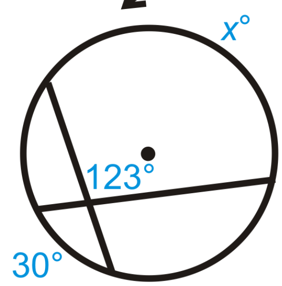
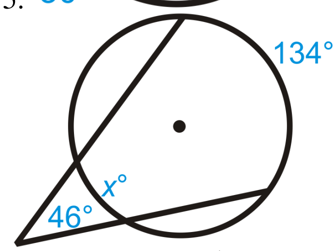
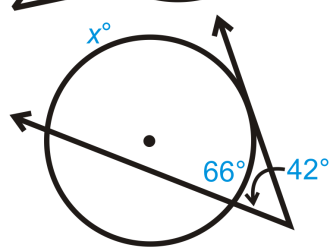


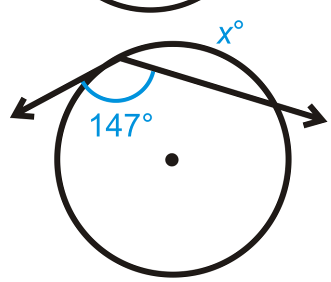
13. 

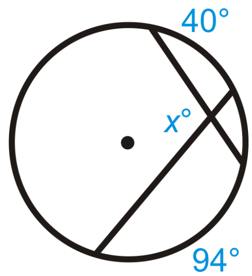

14. 


15. 


16. 


17. 


18. 

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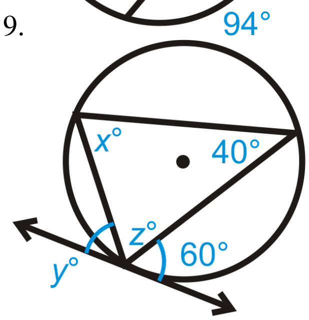


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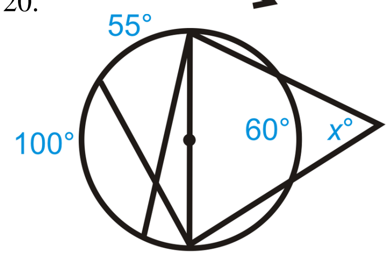


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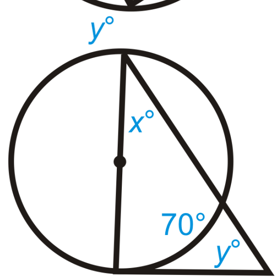


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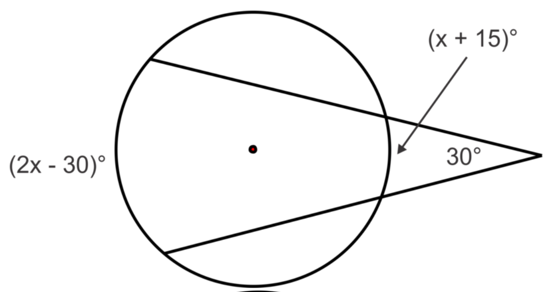
***Algebra Connection*** Solve for the variable(s).

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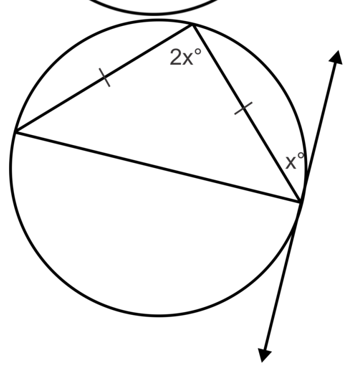


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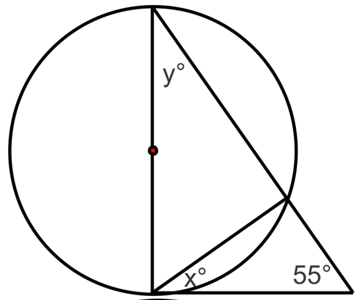


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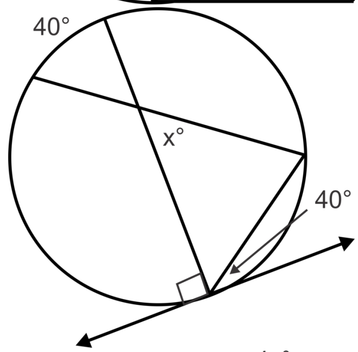


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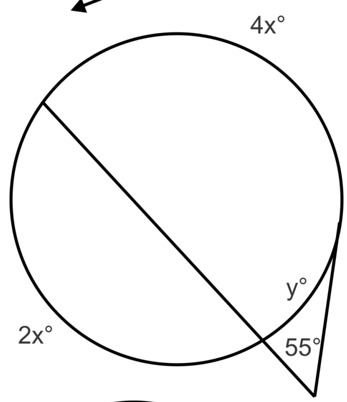


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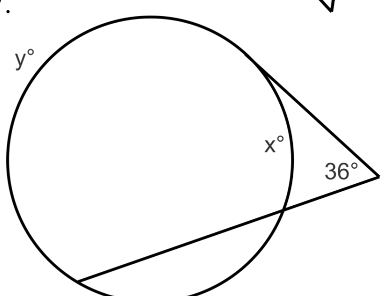


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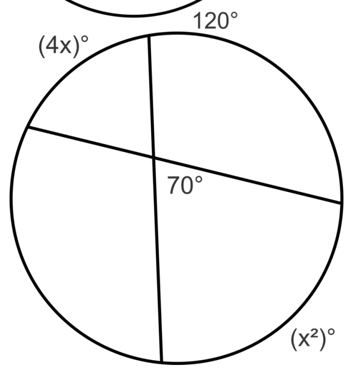
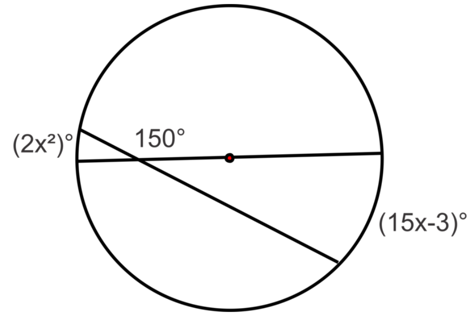
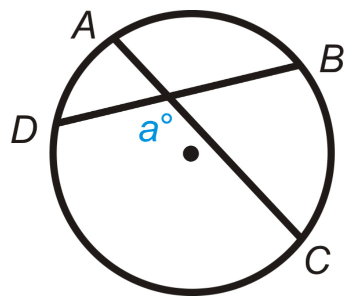




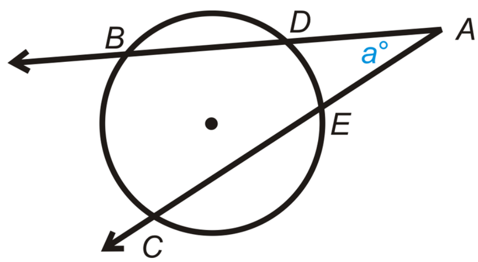

30.

31. Prove Theorem 9-12.




Given: Intersecting chords $\overline{AC}$ and $\overline{BD}$. Prove: $m\angle a = \frac{1}{2}\left(m\widehat{DC} + m\widehat{AB}\right)$ *HINT:* Draw $\overline{BC}$ and use inscribed angles.

32. Prove Theorem 9-13.




Given: Secant rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ Prove: $m\angle a = \frac{1}{2}\left(m\widehat{BC} - m\widehat{DE}\right)$ *HINT:* Draw $\overline{BE}$ and use inscribed angles.

## Review Queue Answers

1. $m\angle OML = m\angle OPL = 90^\circ$ because a tangent line and a radius drawn to the point of tangency are perpendicular.
2. $$\begin{aligned} 165^\circ + m\angle OML + m\angle OPL + m\angle MLP &= 360^\circ \\ 165^\circ + 90^\circ + 90^\circ + m\angle MLP &= 360^\circ \\ m\angle MLP &= 15^\circ \end{aligned}$$
3. $m\widehat{MNP} = 360^\circ - 165^\circ = 195^\circ$
4. $\frac{195^\circ - 165^\circ}{2} = \frac{30^\circ}{2} = 15^\circ$, this is the same as $m\angle MLP$.